The Salter Sink: Variable Density Turbulent Incompressible Flow

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Abstract

We simulate variable density turbulent incompressible flow by the G2 finite element method in a study of a projected device for mixing warm ocean surface water into cooler deeper water for the purpose of locally reducing the surface temperature to prevent the formation of hurricanes.

1 General Circulation Models

Global and local climate simulation is based on computational solution of a system of partial differential equations describing the coupled ocean/atmosphere circulation of water/air driven by Solar radiation and Coriolis forces from the rotation of the Earth, in a *General Circulation Model GCM* [1].

The basic GCM for the coupled ocean/atmosphere is the *Navier-Stokes equations* for the *turbulent flow* of water/air viewed as a fluid of *variable density* and *small viscosity*, which is *incompressible* in water and *compressible* in air, combined with a *transport equation* for *salinity* in the ocean and *moisture* in the atmosphere [3], [4].

In this note we report on computational simulation of a projected device consisting of a vertical tube immersed in the ocean with top inlet of warm surface water and bottom outlet in cooler deeper water, driven by incoming waves, for the purpose of preventing formation of hurricanes by lowering the temperature of the surface water, referred to as the *Salter Sink*.

2 The Navier-Stokes Equations as GCM

We describe coupled ocean/water circulation by the *Navier-Stokes equations* for a slightly viscous fluid of variable density filling the volume Ω in $\mathbb{R}^3$ occupied by water/air with boundary Γ representing the seafloor, over a time interval $I = [0, \bar{t}]$. The basic dependent variables of the Navier-Stokes equation are the water/air fluid density, velocity u, pressure p, total energy $\epsilon = \rho|u|^2/2 + e$ with ρ density and e heat energy, and salinity/moisture s combined with a constitutive law for density $\rho = \rho(T, S, p)$ given as a function of $T = e/\rho$ temperature and $S = s/\rho$ salinity per unit mass, and also pressure if the fluid is compressible. We assume water to be incompressible and air to be a perfect gas with $p = (\gamma - 1)\rho T$ modulo dependence on S, with γ a gas constant $= 0.4$ for air.

We consider the following system of equations expressing conservation of mass, momentum, total energy and salinity/moisture combined with incompressibility/perfect gas law and boundary/initial values: Find $\hat{u} = (\rho, u, \epsilon, s)$ depending on $(x,t) \in \Omega \cup \Gamma \times I$, such that with $m = \rho u$ momentum and $Q \equiv \Omega \times I$

$$
\begin{array}{rcll}
\dot{\rho} + \nabla \cdot (\rho u) & = & 0 & \text{in } Q, \\
\dot{m} + \nabla \cdot (mu) + \nabla p - \nabla \cdot \sigma - \rho g - f & = & 0 & \text{in } Q, \\
\dot{\epsilon} + \nabla \cdot (\epsilon u + pu) - \rho g \cdot u & = & R & \text{in } Q, \\
\dot{s} + \nabla \cdot (su) & = & 0 & \text{in } Q, \\
\nabla \cdot u = 0 \text{ in water and } p & = & (\gamma - 1)\rho T \text{ in air} & \text{in } Q \\
u_n & = & 0 & \text{on } \Gamma \times I, \\
\sigma_s & = & \beta u_s & \text{on } \Gamma \times I, \\
\hat{u}(\cdot, 0) & = & \hat{u}^0 & \text{in } \Omega,
\end{array}
\tag{1}
$$

where $\dot{u} = \frac{\partial u}{\partial t}$, u_n is the fluid velocity normal to Γ, u_s is the tangential velocity, $\sigma = 2\nu\epsilon(u)$ is viscous stress with $\epsilon(u)$ the usual velocity strain and ν the fluid viscosity, σ_s is the tangential stress, β is boundary skin friction, g is gravitational force per unit mass, f is Coriolis force, R is radiative heat source and $\hat{u}^0$ is a given initial condition. For simplicity, we set heat and salinity/moisture diffusivities to zero, motivated by the fact that these coefficients are small. The skin friction boundary condition models a turbulent boundary layer [2] and can also be used in explicit coupling of ocean and atmosphere.

If we assume also air to be incompressible, which can be motivated for problems on smaller scales, and here leave out dependence on salinity for simplicity, then we can rewrite the system (1) as the Navier-Stokes equations for variable density incompressible flow: Find $\hat{u} = (\rho, u, p)$ such that

$$
\begin{array}{rcll}
\dot{\rho} + (u \cdot \nabla)\rho & = & 0 & \text{in } Q, \\
\rho(\dot{u} + (u \cdot \nabla)u) + \nabla p - \nabla \cdot \sigma - g\rho - f & = & 0 & \text{in } Q, \\
\nabla \cdot u & = & 0 & \text{in } Q, \\
u_n & = & 0 & \text{on } \Gamma \times I, \\
\sigma_s & = & \beta u_s & \text{on } \Gamma \times I, \\
\rho(\cdot, 0) = \rho^0, \quad u(\cdot, 0) & = & u^0 & \text{in } \Omega.
\end{array}
\tag{2}
$$

In this case the temperature $T = e/\rho$ can be computed a posteriori through the following equation for the internal energy e

$$
\dot{e} + (u \cdot \nabla)e = -(\nabla \cdot \sigma) \cdot u + R \text{ in } Q, \tag{3}
$$

where $-(\nabla \cdot \sigma) \cdot u$ is a source of heat from turbulent dissipation, and $T(\cdot, 0)$ is coupled to $\rho(\cdot, 0)$ by a relation e.g. determined by experiment.

3 G2 for Variable-Density Incompressible Flow

We apply the G2 finite element method [2] to (2) with $\beta = \nu = 0$ and $f = 0$ with trial functions being continuous and piecewise linear in space-time, and test function being continuous piecewise linear in space and piecewise constant in time, on a space-time mesh of mesh size h, assuming velocity trial/test-fuctions v satisfy $v \cdot n = 0$ on Γ, refereed to as the cG(1)cG(1) variant of G2. Denoting the corresponding finite element spaces by U_h and V_h respectively GCMG2 takes the form: Find $\hat{u} \in U_h$ with $u(\cdot, 0)$ and $\rho(\cdot, 0)$ given, such that

$$B(\hat{u}, \hat{v}) = 0 \quad \text{for all } \hat{v} \equiv (r, v, q) \in W_h, \tag{4}$$

where

$$\begin{aligned} B(\hat{u}, \hat{v}) = (\rho(\dot{u} + u \cdot \nabla u) + \nabla p - \rho g, v)_Q + (\nabla \cdot u, q)_Q + (\dot{\rho} + u \cdot \nabla \rho, r)_Q \\ +(\delta(\rho u \cdot \nabla u + \nabla p - \rho g), \rho u \cdot \nabla v + \nabla q))_Q + (\delta u \cdot \nabla \rho, u \cdot \nabla r)_Q \end{aligned} \tag{5}$$

with $(\cdot, \cdot)_Q$ appropriate $L_2(Q)$ scalar products and $\delta = h/|u|$ a stabilization parameter.

4 Simulations of Sink Circulation

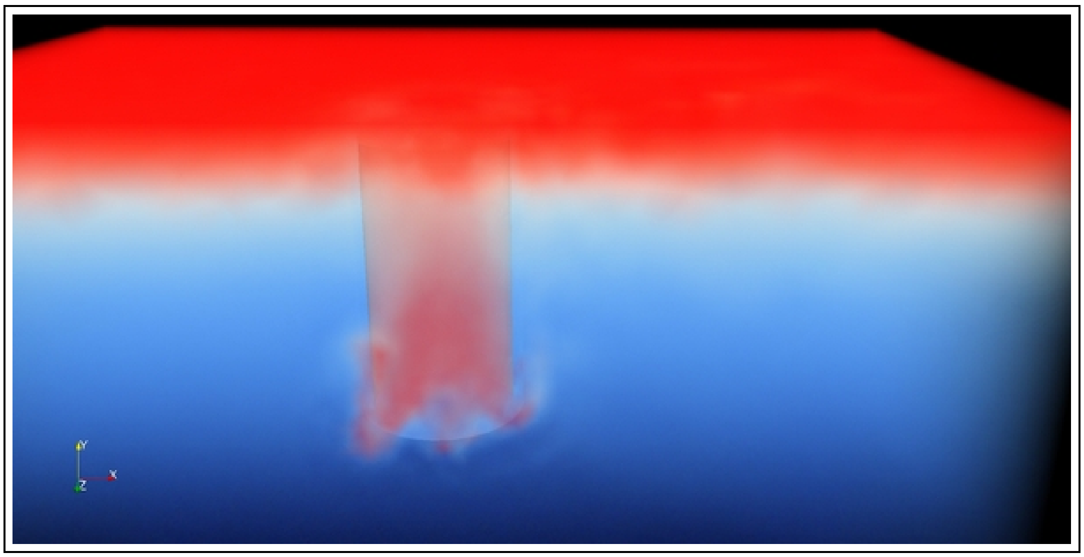

Figure 1: Simulation snapshot of the Salter Sink, showing a flow of warm surface water down the tube to get mixed with cooler deeper water, and then ascending by bouyancy to replace warm surface water with a cooling effect.

The Salter Sink is a device for cooling an ocean surface by mixing warm surface water with deeper cooler water for the purpose of preventing the development of hurricanes. In this study it consists of a vertical tube of length 200 m and diameter 45 m immersed into an ocean of depth 600 m water in which light warm water of temperature 28 degrees Celscius is driven down the tube to get mixed with cooler water at a temperature of 10 degrees at the bottom of the ocean, by an elevated water level inside

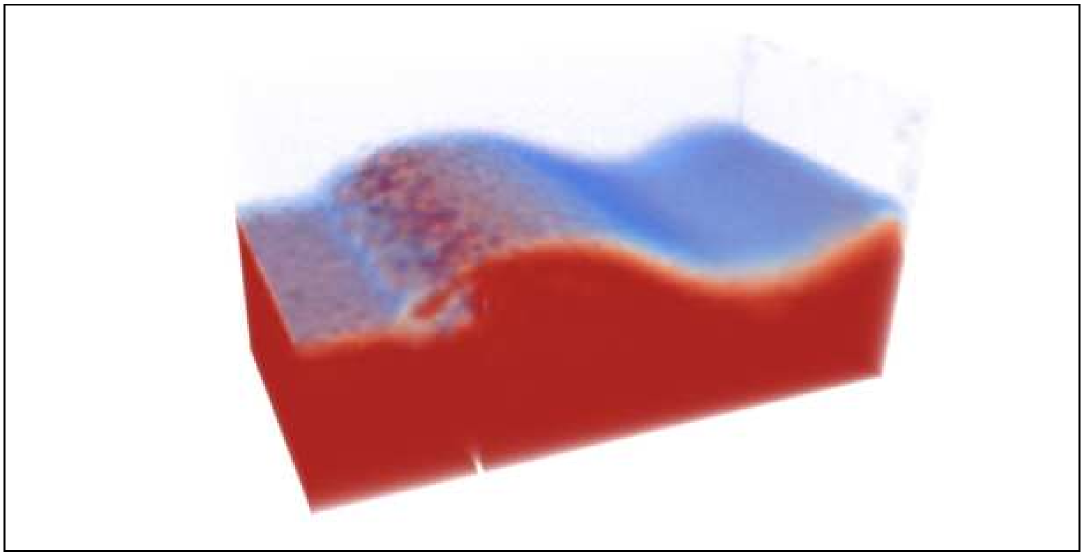

Figure 2: Simulation snapshot of a wave breaking over a wall and elevating the water surface behind the wall.

the tube maintained by incoming water waves. We model the flow computationally by G2 for variable density incompressible Navier-Stokes equations (2) with a given relation between initial temperature and density determined experimentally.

We first simulate the circulation in the sink driven by a vertical force inside the sink modeling the pressure increase from an elevated water surface inside the sink of height 0.15 m and 0.3 m. In this simulation the water surface is fixed and the computation is restricted to the water, as displayed in Fig. 1. The sink is immersed in an ocean current of 0.1 m/s. The temperature variation was determined from the density variation by the experimental relation, instead of a posteriori solving the energy equation as indicated above.

We then simulate the breaking of a water wave as a variable density air-water system as shown in Fig. 2.

The next step is to simulate the complete action of the sink with breaking waves maintaining an elevated water surface inside the sink, and in a further step take also fluid-structure interaction into account into a full simulation including all basic aspects.

References

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