Relationship Between Extremal and Sum Processes Generated By The Same Point Process

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Abstract

We discuss weak limit theorems for a uniformly negligible triangular array (u.n.t.a.) in $Z = [0,\infty) \times [0,\infty)^d$ as well as for the associated with it sum and extremal processes on an open subset $\mathcal{S}$. The complement of $\mathcal{S}$ turns out to be the explosion area of the limit Poisson point process. In order to prove our criterion for weak convergence of the sum processes we introduce and study sum processes over explosion area. Finally we generalize the model of u.n.t.a. to random sample size processes.

Key Words and Phrases: Extremal processes; Increasing processes with independent increments; Weak limit theorems; Levy measure; Poisson point processes; Bernoulli point processes; Random sample size.

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1 Introduction

Collective risk theory basically considers the question about the distribution of the so called total risk process. In order to answer this question the process of the claims met by the insurer is modeled. The relationship between the distribution of the number of claims and the distribution (distributions) of the claim sizes is studied. The usual assumption is that the claims arrive at times $T_k, k = 1, 2, \ldots$, which form a renewal process $N(t), t \geq 0$, i.e. the claim inter-arrival times $Y_k = T_k - T_{k-1}, k \geq 1$ are i.i.d. The claim sizes X_k are positive independent r.v.'s and the sequences $\{T_k\}$ and $\{X_k\}$ are independent. In this framework the total claim amount process is defined by

$$Z(t) = \sum_{k=1}^{N(t)} X_k = \sum_{T_k \leq t} X_k.$$

One of the most important characteristics in this model is namely the distribution $G_t(x) = \mathbf{P}(Z(t) < x)$. It is clear that in general the problem of finding the distribution of a sum of random number independent random variables is not an easy one and often the results are quite complicated. Even in the simple case when $N(t)$ is a Poisson process and the claim sizes X_k are identically distributed having distribution function F we have

$$G_t(x) = \mathbf{P}\left(\sum_{i=1}^{N(t)} X_k < x\right) = \sum_{n=0}^{\infty} \mathbf{P}\left(\sum_{i=1}^{n} X_k < x | N(t) = n\right) \mathbf{P}(N(t) = n)$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{*n}(t).$$

where $F^{*n}(x) = \mathbf{P}(\sum_{i=1}^{n} X_i < x)$ n-fold convolution of F. Due to this fact many authors turn to the approximation of the process $Z(t)$ using another process which finite dimensional distributions are known and more convenient from a computational point of view. The basic idea of such an approximation is to normalize properly the claim arrival times T_k and the claim sizes X_k. For $n \geq 1$ construct $(T_{nk}, X_{nk}) = (\tau_n^{-1}(T_k), u_n^{-1}(X_k))$, where $(\tau_n(t), u_n(x))$ is a regular time-space transformation, i.e. the functions τ_n and u_n are strictly increasing and continuous. [1] In this way when n increases to infinity the number of the claims occurring during a fixed time interval $[0, t]$ gets larger and the claim sizes get smaller. The problem is to find weak limit S for the sequence

$$S_n(t) = \sum_{\tau_n^{-1}(T_k) \leq t} u_n^{-1}(X_k) = \sum_{T_{nk} \leq t} X_{nk}$$

of the transformed total claim amount process. Of course, the norming transformations would have different forms depending on the assumptions for the distributions of Y_k and X_k. Respectively the limiting process S would have different properties. From the formulation of the problem it follows that S shall be self-similar process with independent increments since it appears as a weak limit in uniformly negligible triangular array (u.n.t.a.) $\mathcal{N}_n = \{(T_{nk}, X_{nk})\}, n \geq 1$ which is derived in appropriate way from the sequence (T_k, X_k). Quite interesting is the so called "very-heavy tailed" case when $E(X_k) = \infty$. In this case the norming transformation has the form $u_n(x) = b^{-1}(n)x$, i.e. there is no need of centering but only normalizing the claim sizes. Further in our paper we are interested in the very-heavy tailed case. It is well

[1] In practice usually linear transformations are used.

known that in this case the behavior of the aggregate claim amount is determined by the behavior of the extremal claim amount. On the other hand, the extremal claim amounts $\bigvee_{k=1}^{N(t)} X_k = \bigvee_{T_k \le t} X_k$ are also of great importance in the collective risk theory. Thus, the relationship between the convergence of extremal and sum processes generated by the same u.n.t.a. will be investigated.

Since the paper deals with processes and functions taking values in $\mathbb{R}^d, d > 1$ we introduce the following notations.

The non random $d-$dimensional vectors will be denoted by $\mathbf{x} = (x^{(1)}, \ldots, x^{(d)})$.

The random vectors and processes with phase space $\mathbb{R}^d$ will be denoted by $\mathbf{U} = (U^{(1)}, \ldots, U^{(d)})$ and $\mathbf{Y}(t) = (Y^{(1)}(t), \ldots, Y^{(d)}(t)), \quad t \in [0, \infty)$, respectively. The inequalities between two vectors

$$\mathbf{x} < \mathbf{y}, \quad (\mathbf{x} \le \mathbf{y})$$

mean

$$x^{(i)} < y^{(i)} \quad (x^{(i)} \le y^{(i)}) \quad \text{for all } i = 1, 2, \ldots, d.$$

The operation "maximum" $\vee$ between two vectors has to be read as

$$\mathbf{x} \vee \mathbf{y} = (x^{(1)} \vee y^{(1)}, x^{(2)} \vee y^{(2)}, \ldots, x^{(d)} \vee y^{(d)}).$$

The same for the operation "addition"

$$\mathbf{x} + \mathbf{y} = (x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \ldots, x^{(d)} + y^{(d)}).$$

We will also use the vectors $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. For any two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \le \mathbf{b}$ we define the closed interval $[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} : \mathbf{a} \le \mathbf{x} \le \mathbf{b}\}$. The open and half open intervals are defined in a similar way.

Let

$$\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \ge 1\}, \quad n \ge 1 \tag{1.1}$$

be a sequence of time-space point processes. Further we suppose that for every $n = 1, 2, \ldots$, the point process $\mathcal{N}_n$ is defined on the open (hence locally compact) subset $\mathcal{S}_n$ of $[0, \infty) \times [0, \infty)^d$. The time points $t_{nk} \in [0, \infty)$ are distinct and non-random, and $\mathbf{X}_{nk}$ are row-wise independent random vectors in $[0, \infty)^d$.

We assume that

$$\mathcal{N}_n([0, t] \times [\mathbf{0}, \mathbf{x})^c) < \infty \text{ a.s.} \tag{1.2}$$

if $[0, t] \times [\mathbf{0}, \mathbf{x})^c \subset \mathcal{S}_n$.

Remark 1. With an abuse of notation here and later on we denote by $\mathcal{N}_n$ the collection of points (1.1) as well as the random measure $\mathcal{N}_n(A) = \#\{k : (t_{nk}, \mathbf{X}_{nk}) \in A\}, \quad A \subset \mathcal{S}_n$.

The finiteness assumption (1.2) means that almost all realizations of $\mathcal{N}_n$ are finite on compact subsets of $\mathcal{S}_n$.

We associate with $\mathcal{N}_n$ the extremal process

$$\mathbf{Y}_n(t) = \{\vee_k \mathbf{X}_{nk} : t_{nk} \le t\} \tag{1.3}$$

with independent max-increments. One of the main characteristics of any extremal process is its lower curve $\mathbf{C} : [0, \infty) \to [0, \infty)^d$ below which the sample paths of

$\mathbf{Y}$ cannot pass. The lower curve is uniquely determined by the extremal process. Denote by $\mathbf{C}_n$ the lower curve of the extremal process $\mathbf{Y}_n, \quad n \geq 1$. Since $\mathbf{Y}_n(t) \geq \mathbf{C}_n(t)$ a.s. $t \geq 0$, only the points of $\mathcal{N}_n$ which belong to $[\mathbf{0}, \mathbf{C}_n]^c \subset \mathcal{S}_n$ contribute to the behavior of the extremal process $\mathbf{Y}_n$. Here $[\mathbf{0}, \mathbf{C}_n] = \bigcup_t [\mathbf{0}, \mathbf{C}_n(t)]$ is the set below the lower curve $\mathbf{C}_n$.

Define the sum process

$$\mathbf{S}_n(t) = \{\sum_k \mathbf{X}_{nk} : (t_{nk}, \mathbf{X}_{nk}) \in [\mathbf{0}, \mathbf{C}_n]^c, t_{nk} \leq t\}$$

with independent additive increments. Therefore, the values of $\mathbf{S}_n$ and $\mathbf{Y}_n$ are determined by the same points of $\mathcal{N}_n$ which belong to $[\mathbf{0}, \mathbf{C}_n]^c$.

Having the three sequences:

- the sequence of point processes $\mathcal{N}_n$;
- the sequence of extremal processes $\mathbf{Y}_n$;
- the sequence of sum processes $\mathbf{S}_n$,

we investigate the following problems:

- the convergence of each sequence under appropriate normalization;
- the properties of the limiting processes;
- the relationships between the convergence of the three sequences.

The paper is organized as follows. In Section 2 we investigate the convergence of the sequence of point processes $\mathcal{N}_n$ and the sequence of extremal processes $\mathbf{Y}_n$. The main result of the section is Theorem 2 which establishes the relation between the vague convergence the sequence $\mathcal{N}_n$ and the weak convergence of the sequence $\mathbf{Y}_n$. The section also contains some basic results for extremal processes obtained by Balkema and Pancheva [1] which are needed in the next sections.

In Section 3 we prove the decomposition and the representation for the characteristic function of a stochastically continuous sum process $\mathbf{S}$, defined above the lower curve of a given extremal process $\mathbf{Y}$.

In Section 4 we study the relation between the weak convergence of the sequences $\mathbf{Y}_n$ and $\mathbf{S}_n$. The main result is Theorem 5, the Functional Extremal Criterion for the weak convergence of the sequence $\mathbf{S}_n$ to the stochastically continuous limiting process $\mathbf{S}$ considered in Section 3.

In Sections 2, 3, and 4 we assume that the time points of the point processes $\mathcal{N}_n$ are deterministic. In the last Section 5 we generalize the model of u.n.t.a. and consider a sequence of Bernoulli point processes $\mathcal{N}_n$ with random time components T_{nk}. Thus, the associated sum process $\mathbf{S}_n(t)$ and extremal process $\mathbf{Y}_n(t)$ are of random sample size $N_n(t) = \max\{k : T_{nk} \leq t\}$. In this section we give conditions for the weak convergence of $\mathcal{N}_n$ to a Cox process $\tilde{\mathcal{N}}$ (Theorem 6) and also weak convergence of $\mathbf{Y}_n$ and $\mathbf{S}_n$ to a compound extremal - and a compound sum process, respectively (Theorem 7 and 8). A special case of triangular array $\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ is considered in Pancheva and Jordanova (2004a): $t_{nk} = \tau_n^{-1}(t_k)$ and $\mathbf{X}_{nk} = U_n^{-1}(\mathbf{X}_k)$ where $\eta_n(t, \mathbf{x}) = (\tau_n(t), U_n(\mathbf{x}))$ is a coordinate-wise strictly increasing continuous mapping and $\mathbf{X}_k, k \geq 1$ are i.i.d. random vectors in $[0, \infty)^d$. The corresponding limit extremal process is max-stable. Thus, its lower curve is constant, say $\mathbf{C}(t) \equiv 0$, and one does not observe explosion area phenomena. A particular functional extremal criterion is proved there. The random sample size generalization is studied later on in Pancheva and Jordanova (2004b).

2 Relationship between an Extremal Process and the underlying Bernoulli Point Process

In this section we make a survey of some published and some unpublished results obtained by Balkema and Pancheva during their collaboration in 1995 - 2000. We shall refer to their paper (1996) as $BP'96$ for brevity. In $BP'96$ a point process $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ on an open subset $\mathcal{S}$ of $Z = [0,\infty) \times [0,\infty)^d$ is called **Bernoulli point process** (B.p.p.) if

(a) Its mean measure (m.m.) $\mu(\cdot) = \mathbf{E}\mathcal{N}(\cdot)$ is a Radon measure on $\mathcal{S}$ (i.e. it is finite on compact subsets of $\mathcal{S}$);

(b) $\mathcal{N}$ is simple in time, i.e. $T_i \neq T_j$ for $i \neq j$;

(c) For any integer m the restrictions $\mathcal{N}_1, \ldots, \mathcal{N}_m$ of $\mathcal{N}$ to (time) slices over disjoint time intervals $I_1, \ldots, I_m$ are independent.

As a first example of B.p.p. one can take a simple in time Poisson point process. Bernoulli point processes are important for the study of the so called extremal processes.

An **extremal process** $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ is a random process with right-continuous increasing sample paths and independent max-increments. More precisely, for any finite sequence of time points $0 = t_0 < t_1 < \ldots < t_m$ there exist independent random vectors $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_m$ such that

$$(\mathbf{Y}(t_0), \ldots, \mathbf{Y}(t_m)) \stackrel{d}{=} (\mathbf{U}_0, \mathbf{U}_0 \vee \mathbf{U}_1, \ldots, \mathbf{U}_0 \vee \ldots \vee \mathbf{U}_m)$$

The main **characteristics** of an extremal process $\mathbf{Y}$ (cf $BP'96$) are its:

- **lower curve** $\mathbf{C} : [0,\infty) \to [0,\infty)^d$, increasing and right continuous, below which the sample paths of $\mathbf{Y}$ cannot pass. It is defined coordinate-wise: $C^{(i)}(t)$ is the left endpoint of the distribution function (df) of the i-th coordinate of the random vector $\mathbf{Y}(t)$, $\quad i = 1, \ldots, m$. Thus, $\mathbf{Y}(t) \geq \mathbf{C}(t)$, a.s. The lower curve is uniquely determined by the process $\mathbf{Y}$;

- **distribution function** $f : (0,\infty)^{d+1} \to [0,1]$, $\quad f(t,\mathbf{x}) = \mathbf{P}(\mathbf{Y}(t) < \mathbf{x})$. It is decreasing and right-continuous in t and increasing and left continuous in $\mathbf{x}$, hence lower semi-continuous. The family $F_t(\cdot) = f(t,\cdot), t \geq 0$ of the univariate marginal distributions of $\mathbf{Y}$ determines uniquely the finite dimensional distributions (f.d.d.) of $\mathbf{Y}$, hence the process itself. More precisely, for $t_1 < \ldots < t_n$ in $(0,\infty)$ and $\mathbf{x}_1 < \ldots < \mathbf{x}_n$ in $(0,\infty)^d$

$$F_{t_1,\ldots,t_n}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \begin{cases} 0, & \text{if } \min_i F_{t_i}(\mathbf{x}_i) = 0 \\ F_{t_1}(\mathbf{x}_1)\frac{F_{t_2}(\mathbf{x}_2)}{F_{t_1}(\mathbf{x}_2)} \cdots \frac{F_{t_n}(\mathbf{x}_n)}{F_{t_{n-1}}(\mathbf{x}_n)}, & \text{otherwise} \end{cases}$$

- **max-increments** $\mathbf{U}(s,t]$ over time intervals $(s,t]$, $\quad 0 \leq s < t, \quad t > 0$. The Structure Theorem in $BP'96$ states that for any extremal process $\mathbf{Y}$ there exists a consistent family of max-increments $\mathbf{U}(s,t]$ (assuming the underlying probability space is sufficiently rich) so that

(i) $\mathbf{U}(s,t] \geq \mathbf{C}(t), \quad 0 \leq s < t;$

(ii) $\mathbf{Y}(t) = \mathbf{Y}(s) \vee \mathbf{U}(s,t], \quad 0 \leq s < t;$

(iii) for disjoint time intervals $I_1, \ldots, I_m$ the random vectors $\mathbf{U}(I_1), \ldots, \mathbf{U}(I_m)$ are independent.
- **underlying Bernoulli point process** $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ on the open set $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ such that $\mathbf{Y}$ can be represented as

$$\mathbf{Y}(t) = \mathbf{C}(t) \vee \{\vee \mathbf{X}_k : T_k \leq t\} \tag{2.1}$$

Here $\mathbf{X}_k$ are independent random vectors in $[0,\infty)^d$ and T_k are distinct random time points such that $(T_k, \mathbf{X}_k) \in \mathcal{S}$ a.s. In the presence of a lower curve, i.e. $\mathbf{C}(t) \neq 0$ for some $t \geq 0$, the distribution of the p.p. $\mathcal{N}$ is not uniquely determined by the distribution of the associated extremal process $\mathbf{Y}$. This holds even for a Poisson p.p. associated with a max-id extremal process. So, different B.p.p.'s on $\mathcal{S}$ may generate the same extremal process $\mathbf{Y}$ by (2.1). Later in this section we shall consider closely this form of the phenomenon **blotting**, discussed in $BP'96$ and related to different sources of lack of uniqueness in Extreme Value Theory.

EXAMPLE 2.1. Assume that a p.p. $\mathcal{N}$ on $[0,1] \times (0,1)$ consists of two points $(1/3, X_1)$ and $(3/4, X_2)$ where X_1 and X_2 are independent r.v.'s, X_1 is uniformly distributed in $(0,1/2)$ and X_2 is uniformly distributed in $(1/2,1)$. The extremal process Y, generated by $\mathcal{N}$,

$$Y(t) = \begin{cases} 0, & t \in [0,1/3) \\ X_1, & t \in [1/3,3/4) \\ X_2, & t \in [3/4,1) \end{cases} \tag{2.2}$$

has discontinuous lower curve $C(t)$

$$C(t) = \begin{cases} 0, & t \in [0,3/4) \\ 1/2, & t \in [3/4,1) \end{cases} \tag{2.3}$$

□

EXAMPLE 2.2. Let $\mathcal{N} = \{(t_k, \mathbf{X}_k) : k \geq 1\}$ be a point process on $(0,\infty) \times [0,\infty)^d$ where t_k are distinct non-random time points, increasing to ∞, and $\{\mathbf{X}_k\}$ are independent random vectors on $[0,\infty)^d$. Define $k(t) = \#\{k : t_k \leq t\}$. Then $\mathcal{N}$ is Bernoulli and the lower curve of the extremal process $\mathbf{Y}(t) = \bigvee_{k=1}^{k(t)} \mathbf{X}_k$ is identically zero.

□

EXAMPLE 2.3. Let $\mathbf{C} : [0,\infty] \to [0,\infty)^d$ be an increasing and right-continuous curve. Let $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ be a Bernoulli p.p. on the open set $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$. Then $\mathcal{N}([0,t] \times [\mathbf{0}, \mathbf{x})^c) < \infty$ a.s. for all $t \geq 0, \mathbf{x} > \mathbf{C}(t)$. Assume the corresponding counting process

$$N(t) = \max\{k : T_k \leq t\}$$

is finite for every finite t. Then

$$\mathbf{Y}(t) := \mathbf{C}(t) \vee \bigvee_{k=1}^{N(t)} \mathbf{X}_k$$

is an extremal process.

□

We call an extremal process **stochastically continuous** at a point $t > 0$ if

$$\mathbf{Y}(t) = \mathbf{C}(t) \vee \mathbf{Y}(t-0) \text{ a.s.} \tag{2.4}$$

Thus, a stochastically continuous extremal process does not have fixed discontinuity points except possible discontinuities of the lower curve.

The Decomposition Theorem in $BP'96$ states that an extremal process with lower curve $\mathbf{C}$ and underlying B.p.p. $\mathcal{N}$ can be decomposed as the maximum

$$\mathbf{Y}(t) = \mathbf{Y}_c(t) \vee \mathbf{Y}_d(t)$$

of two independent extremal processes $\mathbf{Y}_c$ and $\mathbf{Y}_d$ with common lower curve $\mathbf{C}$. The process $\mathbf{Y}_c$ satisfies (2.4) for all $t > 0$ and $\mathbf{Y}_c(0) = \mathbf{C}(0)$, a.s. It is associated with a Poisson p.p. $\mathcal{N}^c$ on $\mathcal{S}$ which m.m. μ does not charge any instant sections $\mathcal{S}(t)$ of $\mathcal{S}$, i.e. $\mu(\mathcal{S}(t)) = 0$, for all $t \geq 0$. The associated with $\mathbf{Y}_d$ B.p.p. $\mathcal{N}^d$ is the sum of zero-one p.p.'s $\mathcal{N}_k = (t_k, \mathbf{X}_k)$ where $t_k \geq 0$ are points for which (2.4) fails to hold, and $\mathbf{X}_k \geq \mathbf{C}(t_k)$ are independent random vectors.
Recall, an extremal process is max-id if for all $n > 1$ there are n i.i.d. random processes $\mathbf{Y}_{n1}, \ldots, \mathbf{Y}_{nn}$ such that $\mathbf{Y} = \mathbf{Y}_{n1} \vee \ldots \vee \mathbf{Y}_{nn}$. The most studied relationship between an extremal process $\mathbf{Y}$ with lower curve $\mathbf{C}$ and its underlying B.p.p. $\mathcal{N}$ on $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ is in the max-id case: $\mathbf{Y}$ is max-id iff $\mathcal{N}$ is Poisson. There is a close relation between the df $f(t, \mathbf{x}) = \mathbf{P}(\mathbf{Y}(t) < \mathbf{x})$ of a max-id extremal process and the mean measure μ of the underlying Poisson p.p., namely

$$f(t,\mathbf{x}) = \begin{cases} \exp\{-\mu([0,t] \times [\mathbf{0}, \mathbf{x})^c)\}, & \mathbf{x} > \mathbf{C}(t),\ t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The closed set $[\mathbf{0}, \mathbf{C}]$ below the lower curve is an "explosion area" for the mean measure μ, i.e. $\mu = \infty$ there as $f(t, \mathbf{x}) = 0$ for all $(t, \mathbf{x}) \in [\mathbf{0}, \mathbf{C}]$. Above the lower curve, on the set $\mathcal{A}$ the measure μ is finite as $f(t, x)$ is positive for $\mathbf{x} > \mathbf{C}(t),\ t \geq 0$. In the univariate case there are no other areas. In the multivariate case the underlying p.p. $\mathcal{N}$ is defined on the open set $[\mathbf{0}, \mathbf{C}]^c$. One can not easily blot out the points of $\mathcal{N}$ in both sandwich areas between $[\mathbf{0}, \mathbf{C}]$ and $\mathcal{A}$: they may contribute to the mass of f on $\mathcal{A}$. So, different p.p.'s on $\mathcal{S}$ may generate the same extremal process. We meet here another form of the **blotting** phenomenon.

Following Kallenberg (1997), a p.p. $\mathcal{N}$ is usually defined on a locally compact separable metric space. In our case the set $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ is open, hence locally compact. Let $\mathcal{N}_n$ be a sequence of Bernoulli point processes on $\mathcal{S}$. We say that $\mathcal{N}_n$ is **vaguely convergent** to a p.p. $\mathcal{N}$ on $\mathcal{S}$, briefly $\mathcal{N}_n \stackrel{v}{\to} \mathcal{N}$, if for any relatively compact subset $K \subset \mathcal{S}$ with $\mathbf{P}(\mathcal{N}(\partial K) = 0) = 1$ the convergence $\mathcal{N}_n(K) \stackrel{d}{\to} \mathcal{N}(K)$ holds (cf Resnick (1987)). Unfortunately, the limit p.p. $\mathcal{N}$ may be problematic:
- the convergence $\stackrel{v}{\to}$ gives no information about the behavior of $\mathcal{N}$ on the lower curve. There the m.m. μ may be finite or infinite;

- space points of $\mathcal{N}$ may be even in the interval $(\mathbf{C}(t_0 - 0), \mathbf{C}(t_0))$ when the lower curve $\mathbf{C}$ of the associated extremal process $\mathbf{Y}$ is discontinuous at t_0. By the Structure Theorem, $\mathbf{Y}(t) = \mathbf{Y}(t-0) \vee \mathbf{U}(t)$ where $\mathbf{U}(t) = \lim_n \mathbf{U}(s_n, r_n]$ a.s. for $s_n \uparrow t$ and $r_n \downarrow t$;

- time points may cluster to one point in the limit. In this case the limit p.p. is no more B.p.p.

EXAMPLE 2.4. Let $\mathcal{N}_1$ and $\mathcal{N}_{2,n}$, $\quad n = 1, 2, \ldots$ be simple in time Poisson p.p.'s lying respectively on $[0, \infty)^2$ and on the line $x(t) = (t - t_0) \tan \alpha_n$, $\quad \alpha_n \in (0, \pi/2), t \geq 0$ through the point t_0. Assume $\alpha_n \uparrow \pi/2$ as $n \to \infty$. Then the superposition $\mathcal{N}_n = \mathcal{N}_1 \oplus \mathcal{N}_{2,n}$ converges: $\mathcal{N}_n \xrightarrow{v} \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ where $\mathcal{N}_2$ lies on the instant space through t_0. As a limit of Poisson p.p.'s $\mathcal{N}$ is Poisson but not Bernoulli. Its m.m. μ charges the instant section through t_0.

□

For a convergence of B.p.p.'s the following statement applies.

Theorem 1. *Suppose $\mathcal{N}_n$ is a Bernoulli point process on an open subset $\mathcal{S} \subset Z$ with mean measure μ_n, for $n \geq 1$. Let μ be a Radon measure on $\mathcal{S}$. If*

$$\mu_n \xrightarrow{v} \mu \text{ on } \mathcal{S} \tag{i}$$

and

$$\sup\{\mu_n(K(s)) : 0 \leq s \leq t\} \to 0 \text{ as } n \to \infty \tag{ii}$$

for every $t > 0$ and every relatively compact subset $K \subset \mathcal{S}$, then the sequence $\mathcal{N}_n$ converges vaguely to a Poisson p.p. $\mathcal{N}$ on $\mathcal{S}$ with mean measure μ. Here $K(s)$ is the instant section of the set K.

Proof. Let us fix an arbitrary compact set $K \subset \mathcal{S}$. For all n we decompose

$$\mathcal{N}_n(K) = \mathcal{N}_n^c(K) \oplus \mathcal{N}_n^d(K) \tag{2.5}$$

where $\mathcal{N}_n^c$ is a Poisson p.p. and $\mathcal{N}_n^d$ is a sum of independent zero-one p.p.'s $\mathcal{N}_{nk}$ on instant spaces $K(a_{nk})$ of positive measure, $\mu_n(K(a_{nk})) > 0$. The set $A_n = \{a_{n1}, a_{n2}, \ldots\}$ is at most countable and $\sum_k \mu_n(K(a_{nk})) \leq \mu_n(K)$ is finite.

In fact, every zero-one p.p. $\mathcal{N}$ can be embedded in a Poisson p.p. $R := \{\mathbf{X}_1, \ldots, \mathbf{X}_L\}$ where L is a Poisson r.v. with $\mathbf{E}L = \lambda$ and $\mathbf{X}_1, \mathbf{X}_2, \ldots$ are i.i.d. random variables, so that

$$\mathcal{N} = \begin{cases} \emptyset, & \text{if } L = 0 \\ \{\mathbf{X}_1\}, & \text{otherwise.} \end{cases}$$

Then $\mathbf{P}(\mathcal{N} \neq R) = \mathbf{P}(L \geq 2) = 1 - e^{-\lambda} - \lambda e^{-\lambda} \leq \lambda(1 - e^{-\lambda}) \leq \lambda^2$.
On the other hand $\mu = \mathbf{E}\mathcal{N} = \mathbf{P}(|\mathcal{N}| = 1) = \mathbf{P}(L > 0) = 1 - e^{-\lambda}$, where $|\mathcal{N}|$ is the number of points of $\mathcal{N}$. Hence for $\mu \to 0$, $\lambda^2 = (\log(1-\mu))^2 \sim \mu^2$. Thus, we can replace the zero-one p.p.'s $\mathcal{N}_{nk}$ by independent Poisson p.p.'s R_{nk}, so that

$$\mathbf{P}(\mathcal{N}_{nk} \neq R_{nk}) \leq \mu_n^2(K(a_{nk})) \to 0, \text{ as } n \to \infty$$

Then $R_n := \sum_k R_{nk}$ is a Poisson p.p. on K and

$$\begin{aligned} &\mathbf{P}(R_n(K) \neq \mathcal{N}_n^d(K)) \leq \textstyle\sum_k \mathbf{P}(R_{nk} \neq \mathcal{N}_{nk}) \\ &\leq \textstyle\sum_k \mu_n^2(K(a_{nk})) \leq \sup_k \mu_n(K(a_{nk})) \sum_k \mu_n(K(a_{nk})) \to 0 \end{aligned}$$

in view of (ii) and the finiteness of $\mu(K)$. Now we may replace $\mathcal{N}_n^d$ by R_n in the above decomposition (2.5) of $\mathcal{N}_n$. It is well known that Poisson p.p.'s converge vaguely iff their m.m.'s converge vaguely. The limit process $\mathcal{N}$ is then Poisson, too, but eventually not simple in time.
The convergence $\mathcal{N}_n \xrightarrow{v} \mathcal{N}$ holds on $\mathcal{S}$ since it holds on any compact subset K of $\mathcal{S}$. □

Remark 2. By Theorem 14.16 in Kallenberg (1997), if $\mathcal{N}$ is a simple p.p. the convergence $\mathcal{N}_n \xrightarrow{v} \mathcal{N}$ is equivalent to the weak convergence $\mathcal{N}_n \Rightarrow \mathcal{N}$.

Below we discuss the **question** whether the vague convergence of Bernoulli p.p.'s $\mathcal{N}_n$ always imply weak convergence of the associated extremal processes $\mathbf{Y}_n$, supposing additionally that the limit p.p. $\mathcal{N}$ is simple in time.

Let us first recall when a sequence of extremal processes is weakly convergent. Given a sequence of extremal processes $\{\mathbf{Y}_n\}, \mathbf{Y}_n : [0, \infty) \to [0, \infty)^d$ we denote the distribution function and the probability distribution (p.d.) of $\mathbf{Y}_n$ on $\mathcal{M}([0, \infty))$ by f_n and π_n respectively. For fixed $t > 0$ let $F_{nt}(\cdot) = f_n(t, \cdot)$. We say the sequence $\{\mathbf{Y}_n\}$ is weakly convergent to an extremal process $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ with df f and p.d. π, briefly $\mathbf{Y}_n \Rightarrow \mathbf{Y}$, if one of the following equivalent statements holds (cf Th. 1,§ 6 in $BP'96$)

(1) $f_n \to f$ at all continuity points of f;

(2) $F_{nt} \to F_t = f(t, \cdot)$ weakly for each t in a dense subset of $(0, \infty)$;

(3) $\int \phi d\pi_n \to \int \phi d\pi$ for bounded $\phi : \mathcal{M}([0, \infty)) \to \mathbb{R}$ which are continuous in the weak topology of $\mathcal{M}([0, \infty))$.

Now we come back to the above **question**. The following example shows a possible pitfall.

EXAMPLE 2.5. Assume that p.p.'s $\mathcal{N}_n = \mathcal{N}$, for all $n \geq 1$ where $\mathcal{N}$ is the p.p. defined in EXAMPLE 2.1. and $C_n(t) = C(t)$, for all $n \geq 1$ where $C(t)$ is the lower curve of $\mathcal{N}$ and has the form (2.3). The extremal process $Y(t)$, generated by $\mathcal{N}$ is given by (2.2).

□

Let $C_0(t) \equiv 1/2$ and $\mathcal{N}_0 = \{(3/4, X_2)\}$. Define an extremal process $Y_0(t) = Y(t) \vee C_0(t)$. We observe that $\mathcal{N}_n \xrightarrow{v} \mathcal{N}_0$ on the set $\mathcal{S} = \{(t, x) : t \in (0, 1), x \in (1/2, 1)\}$ but $Y_n \not\Rightarrow Y_0$. Indeed, for the point $(1/3, x)$ with $x \in (0, 1/2)$ one gets

$$f_n(1/3, x) = \mathbf{P}(Y(1/3) < x) = \mathbf{P}(X_1 < x) = x$$

but

$$f_0(1/3, x) = \mathbf{P}(Y_0(1/3) < x) = 0.$$

The following result answers the above **question**.

Theorem 2. *Let* $\mathbf{Y}$ *and* $\mathbf{Y}_n$ *be extremal processes with lower curves* $\mathbf{C}$ *and* $\mathbf{C}_n$*, respectively. Let* $\mathcal{N}$ *and* $\mathcal{N}_n$ *be the underlying Bernoulli p.p.'s defined on* $[\mathbf{0}, \mathbf{C}]^c$ *and* $[\mathbf{0}, \mathbf{C}_n]^c$*, respectively. Suppose that* $\mathbf{C}$ *satisfies the following lower curve condition*

$$\begin{aligned} &\mathbf{C}_n \vee \mathbf{C} \to \mathbf{C} \text{ weakly on } (0, \infty) \text{ and} \\ &\lim_n f_n^{(i)}(t, x) = 0 \text{ for all } x < C^{(i)}(t - 0), t > 0, i = 1, \ldots, d. \end{aligned} \tag{LC}$$

If $\mathcal{N}_n \xrightarrow{v} \mathcal{N}$ *on* $[\mathbf{0}, \mathbf{C}]^c$ *then* $\mathbf{Y}_n \Rightarrow \mathbf{Y}$.

Remark 3. In Theorem 1 we have assumed that the Bernoulli p.p.'s $\mathcal{N}_n$ and the limit p.p. $\mathcal{N}$ are defined on the same space $\mathcal{S}$. Theorem 1 remains true if every $\mathcal{N}_n$ is defined on another (locally compact) space $\mathcal{S}_n$ in such a way, that every point $z \in \mathcal{S}$ has a neighborhood U which is contained in all $\mathcal{S}_n$, for $n \geq n_0(U)$. The first part of

(LC) implies that for every compact set $K \subset \mathcal{S}$ there is a number n_0 so that $K \subset \mathcal{S}_n$, for all $n > n_0$.

Proof. (Theorem 2) Denote by f and f_n the df's of $\mathbf{Y}$ and $\mathbf{Y}_n$, respectively. By Theorem 1, Section 6 in $BP'96$ we have to show that $f_n \to f$ for all continuity points of f. Let $(t, \mathbf{x}) \in \mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ be an arbitrary continuity point. The set $A_{t,\mathbf{x}} = [0, t] \times [\mathbf{0}, \mathbf{x})^c$ belongs to $\mathcal{S}$ whenever $\mathbf{x} > \mathbf{C}(t)$. Then

$$\mathbf{P}(\mathbf{Y}_n(t) < \mathbf{x}) = \mathbf{P}(\mathcal{N}_n(A_{t,\mathbf{x}}) = 0)$$

The set $A_{t,\mathbf{x}}$ is $\mathcal{N}-$continuous a.s., hence

$$\mathcal{N}_n(A_{t,\mathbf{x}}) \stackrel{d}{\to} \mathcal{N}(A_{t,\mathbf{x}})$$

This means $f_n(t, \mathbf{x}) \to f(t, \mathbf{x})$ and consequently $\mathbf{Y}_n \vee \mathbf{C} \Rightarrow \mathbf{Y}$. Finally, we combine the last convergence with the second part of condition (LC) and obtain $\mathbf{Y}_n \Rightarrow \mathbf{Y}$. □

The main conclusion of this section is as follows: given a sequence of B.p.p.'s satisfying the conditions of Theorem 1 with simple in time limit Poisson p.p., the conditions of Theorem 2 guarantee that the sequence of the associated extremal processes is weakly convergent to the max-id extremal process generated by the limit Poisson p.p.

It is very natural to associate with $\mathcal{N}$ a stochastic process $\mathbf{S}$ with independent additive increments, briefly sum process. The following section is devoted to such a process.

3 Sum Process over Explosion Area

Let $\mathbf{C} : [0, \infty) \to [0, \infty)^d, \mathbf{C}(0) = \mathbf{0}$ be an increasing right-continuous curve and $T = \sup\{t \geq 0 : |\mathbf{C}(t)| = 0\}$. Suppose a simple in time Poisson p.p. on Z is given. Its mean measure μ is supposed to be $\sigma-$finite and satisfying the condition

$$\mu(A_{t,\mathbf{x}}) \begin{cases} < \infty, & \mathbf{x} > \mathbf{C}(t),\ t \geq 0 \\ = \infty, & \text{otherwise,} \end{cases} \tag{I}$$

including the case $\mathbf{x} \in (\mathbf{C}(t-0), \mathbf{C}(t+0))$, if t is a discontinuity point of $\mathbf{C}$.

Denote by $\mathcal{S}$ the open set $[\mathbf{0}, \mathbf{C}]^c \subset Z$. The set $[\mathbf{0}, \mathbf{C}]$ is the explosion area of μ. We denote the restriction on $\mathcal{S}$ of the given Poisson p.p. by $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$. Now each point $(T_k, \mathbf{X}_k)$ belongs to $[\mathbf{0}, \mathbf{C}]^c$. In this section we are interested in constructing a sum process $\mathbf{S}$ over the explosion area of μ. Note that $\mathcal{S}$ is not a cone. The case of a cone is considered in Skorokhod (1986), Th. 3.21. Our sum process $\mathbf{S}$ should be a.s. finite, stochastically continuous and having independent increments. Below we define $\mathbf{S}$, decompose it suitably and give the form of its characteristic function. Suppose additionally

$$\mu\{\mathcal{S}(s)\} = 0, \qquad \text{for all } s \in [0, \infty) \tag{II}$$

$$\int_0^{T+\delta} \int_{\{|\mathbf{x}| \leq 1\}} |\mathbf{x}| \mu(ds, \mathbf{dx}) < \infty, \text{ for some } \delta > 0. \tag{III}$$

Condition (I) ensures $\int_0^T \int_{|x| \geq 1} \mu(ds, \mathbf{dx}) < \infty$ thus, (I) and (III) entail

$$\int_0^T \int_{[0,\infty)^d \setminus \{\mathbf{0}\}} (|\mathbf{x}| \wedge 1) \mu(ds, \mathbf{dx}) < \infty.$$

The last condition makes the measure μ a Levy measure. It is well known that every Levy measure is a Radon measure but the converse is not always true. Hence for $t > 0$ one can define the process $\mathbf{S}^{(1)}(t)$ as follows

$$\mathbf{S}^{(1)}(t) = \begin{cases} 0, & t = 0, \\ \sum_{T_k \le t} \mathbf{X}_k, & 0 < t \le T, \\ \mathbf{S}^{(1)}(T), & t > T. \end{cases} \tag{3.1}$$

Indeed, conditions (II) and (III) ensure $\mathbf{P}(|\mathbf{S}^{(1)}(T)| < \infty) = 1$, so definition (3.1) is correct, i.e. the sum on the right hand side converges a.s. Thus, the process $\mathbf{S}^{(1)}$, considered above is a.s. finite, stochastically continuous (in view of (II) and the second part of (I)) with independent increments and its sample paths lie in $\mathcal{S}$. Let us define for arbitrary $h > 0$ and $t \le T$ the process $\mathbf{S}_h^{(1)}(t) = \sum_{T_k \le t} \mathbf{X}_k \mathbb{I}_{\{|\mathbf{X}_k| > h\}}$ and $\mathbf{R}_h^{(1)}(t) = \mathbf{S}^{(1)}(t) - \mathbf{S}_h^{(1)}(t)$. The process $\mathbf{S}_h^{(1)}(t)$ is compound Poisson since it is simply a sum of a.s. finite number of independent r.v's. Both $\mathbf{S}_h^{(1)}(t)$ and $\mathbf{R}_h^{(1)}(t)$ are nonnegative increasing processes ($\mathbf{X}_k \in \mathbb{R}_+^d$). The same holds for $\mathbf{S}_{h_1}^{(1)}(t) - \mathbf{S}_{h_2}^{(1)}(t) \in \mathbb{R}_+^d$ whenever $0 < h_1 < h_2$. As known, each sequence $\mathbf{x}_n \in \mathbb{R}_+^d$, such that $\mathbf{x}_{n-1} - \mathbf{x}_n \in \mathbb{R}_+^d$ is decreasing and bounded from below, hence it converges in $\mathbb{R}_+^d$, i.e. there exists $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}_0 \in \mathbb{R}_+^d$. Therefore $\lim_{h\to 0} \mathbf{R}_h^{(1)}(t) = \mathbf{S}_0^{(1)}(t)$ exists. Here $\mathbf{S}_0^{(1)}(t)$ is a.s. continuous process with independent increments, hence it is a Gaussian process. The increment $\mathbf{S}_0^{(1)}(t) - \mathbf{S}_0^{(1)}(0)$ follows Gaussian distribution and on the other hand it is a.s. positive. This is possible if and only if its variance is equal to zero, which implies $\mathbf{a}(t) := \mathbf{S}_0^{(1)}(t) - \mathbf{S}_0^{(1)}(0)$ is a deterministic increasing continuous function. Since $\lim_{h\to 0} \mathbf{R}_h^{(1)}(t) = \mathbf{a}(t)$ exists and the process $\mathbf{S}^{(1)}(t)$ is a.s. finite, the following decomposition holds

$$\mathbf{S}^{(1)}(t) = \begin{cases} \mathbf{a}(t) + \lim_{h\downarrow 0} \mathbf{S}_h^{(1)}(t), & 0 \le t \le T \\ \mathbf{S}^{(1)}(T), & t > T. \end{cases} \tag{3.2}$$

The characteristic function of $\mathbf{a}(t) + \mathbf{S}_h^{(1)}(t)$ is given by

$$\mathbf{E}e^{i\mathbf{z}.[\mathbf{a}(t)+\mathbf{S}_h^{(1)}(t)]} = \exp\{i\mathbf{z}.\mathbf{a}(t) + \int_0^t \int_{[0,\infty)^d \bigcap \{|\mathbf{x}|>h\}} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\}$$

Here $\mathbf{z}.\mathbf{x}$ means the scalar product of the vectors $\mathbf{z}$ and $\mathbf{x}$. Letting $h \downarrow 0$ in the above equation we get the characteristic function of $\mathbf{S}^{(1)}(t)$

$$\mathbf{E}e^{i\mathbf{z}.\mathbf{S}^{(1)}(t)} = \exp\{i\mathbf{z}.\mathbf{a}(t) + \int_0^t \int_{[0,\infty)^d \setminus \{\mathbf{0}\}} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\}. \tag{3.3}$$

Consider the case $t > T$. It is clear that for $\delta > 0$ and $t \ge T + \delta > T$ if $\int_{T+\delta}^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \mu(ds, \mathbf{dx})$ were infinite then the point process would be infinite a.s. since it is Poisson point process and respectively the sum process would be infinite a.s. too. Since we are interested in a.s. finite case assume the following condition

$$\int_{T+\delta}^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \mu(ds, \mathbf{dx}) < \infty, \text{ for } \delta > 0 \text{ and } t \ge T + \delta. \tag{IV}$$

Condition (IV) combined with (III) provides that the sum process

$$\mathbf{S}^{(2)}(t) = \begin{cases} 0, & t \le T, \\ \sum_{T<T_k\le t} \mathbf{X}_k, & t > T. \end{cases} \tag{3.4}$$

is a.s. finite (cf Theorem 10.15 in Kallenberg (1997)), and can be decomposed as

$$\mathbf{S}^{(2)}(t) = \begin{cases} \mathbf{a}^* + \lim_{\delta\downarrow 0} \mathbf{S}^{(2)}(\delta, t), & T < t \\ 0, & 0 \le t \le T, \end{cases} \tag{3.5}$$

with

$$\mathbf{S}^{(2)}(\delta, t) = \sum_{T+\delta<T_k\le t} \mathbf{X}_k, \text{ and } \mathbf{a}^* = \lim_{\delta\downarrow 0} \mathbf{S}^{(2)}_\delta,$$

where $\mathbf{S}^{(2)}_\delta = \sum_{T<T_k\le T+\delta} \mathbf{X}_k$. The proof of this decomposition relies on similar arguments as in the first case. The only difference is that the limit is taken along the time and in this way a constant $\mathbf{a}^*$ appears. The characteristic function of $\mathbf{S}^{(2)}(t), t > T$ is given by

$$\mathbf{E}e^{i\mathbf{z}.\mathbf{S}^{(2)}(t)} = \exp\{i\mathbf{z}.\mathbf{a}^* + \int_T^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\}.$$

Define for $t \ge 0$ the process

$$\mathbf{S}(t) = \mathbf{S}^{(1)}(t) + \mathbf{S}^{(2)}(t),$$

or equivalently

$$\mathbf{S}(t) = \begin{cases} \mathbf{S}^{(1)}(t), & 0 \le t \le T \\ \mathbf{S}^{(1)}(T) + \mathbf{S}^{(2)}(t), & t > T. \end{cases}$$

The processes $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are independent.
Moreover, if $\mathbf{C} \equiv \mathbf{0}$ we can put $[\mathbf{0}, \mathbf{C}]^c = [0, \infty)^d \setminus \{\mathbf{0}\}$ so the characteristic function (3.3) of $\mathbf{S}^{(1)}$ can be rewritten as

$$\mathbf{E}e^{i\mathbf{z}.\mathbf{S}^{(1)}(t)} = \exp\{i\mathbf{z}.\mathbf{a}(t) + \int_0^t \int_{[0,\mathbf{C}(s)]^c} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\} \tag{3.6}$$

Finally, the characteristic function of the sum process over explosion area is given by

$$\psi_t(\mathbf{z}) = \mathbf{E}e^{i\mathbf{z}.\mathbf{S}(t)} = \exp\{i\mathbf{z}.\mathbf{a}_T(t) + \int_0^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\}$$

where $\mathbf{a}_T(t)$ satisfies

$$\mathbf{a}_T(t) = \begin{cases} \mathbf{a}(t), & 0 \le t \le T \\ \mathbf{a}(T) + \mathbf{a}^*, & t > T. \end{cases} \tag{3.7}$$

Briefly we write $\psi \sim (\mathbf{a}_T, \mu)$ for the ch.f. of the process $\mathbf{S}$, where the Levy measure of $\mathbf{S}$ is the mean measure of the generating Poisson p.p.
There are two boundary cases. The first is $T = 0$ and then $\mathbf{S}(t)$ is pure jump process with $\mathbf{a}_T(t) = \mathbf{a}^*$. The second is $T = \infty$ then $\mathbf{a}_T(t) = \mathbf{a}(t)$, for all $t \ge 0$. Taking advantage of the results above we have proved the following theorem.

Theorem 3. *Suppose $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ is a simple in time Poisson point process on $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ with mean measure μ, where $\mathbf{C} : [0, \infty) \to [0, \infty)^d, \mathbf{C}(0) = \mathbf{0}$ is increasing and right-continuous. Denote $T = \sup\{t \geq 0 : |\mathbf{C}(t)| = 0\}$ and let μ satisfy conditions (I)-(IV). Then the stochastic process $\mathbf{S}$, defined by $\mathbf{S}(t) = \sum_{T_k \leq t} \mathbf{X}_k, \quad t \geq 0$ is a.s. finite, stochastically continuous and has independent increments. It can be decomposed into a sum of two independent processes $\mathbf{S}(t) = \mathbf{S}^{(1)}(t) + \mathbf{S}^{(2)}(t)$. The process $\mathbf{S}^{(1)}(t)$ is defined by (3.1) and admits decomposition (3.2) and the process $\mathbf{S}^{(2)}(t)$ is defined by (3.4) and admits decomposition (3.5). The characteristic function of $\mathbf{S}$ is given by*

$$\psi_t(\mathbf{z}) = \mathbf{E}e^{i\mathbf{z}.\mathbf{S}(t)} = \exp\{i\mathbf{z}.\mathbf{a}_T(t) + \int_0^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, d\mathbf{x})\}$$

where the function $\mathbf{a}_T(t)$ is defined in (3.7).

Let us remark that the converse statement is also true: If $\mathbf{S}(t)$ is a stochastically continuous sum process on $\mathcal{S}$ with ch.f. $\psi \sim (\mathbf{a}_T, \mu)$, then μ necessarily satisfies conditions (I) - (IV) since these conditions are equivalent to the a.s. finiteness of the sum process.

Remark 4. Each id process can be decomposed into a sum of a stochastically continuous, a deterministic and a discrete process. We consider here only the case of stochastically continuous sum process since it is the most interesting case from a theoretical and practical point of view.

4 Triangular Arrays

Triangular arrays are used in studying the asymptotic behavior of maxima (or sums) of large numbers of r.v.'s. Traditionally the n-th row consists of independent r.v.'s $\mathbf{X}_{n1}, \ldots, \mathbf{X}_{nn}$ and one is interested in the weak limit of the probability distribution of the maxima $\mathbf{X}_{n1} \vee \ldots \vee \mathbf{X}_{nn}$. In order to ensure that the contribution of each separate term in a row to the maxima is small, one imposes a condition of asymptotic negligibility on the individual max-increments $\mathbf{X}_{nk},\ k = 1, \ldots, n$. For sums the negligibility condition is simple: $\mathbf{X}_{nk} \to 0$ in probability as $n \to \infty$, uniformly in k. For maxima the condition may depend on the limit: If $\mathbf{X}_{11} \vee \ldots \vee \mathbf{X}_{nn} \xrightarrow{d} \mathbf{X}$ and $\mathbf{X}$ has lower endpoint q, then the asymptotic negligibility condition says $\max_k\{1 - \mathbf{P}(\mathbf{X}_{nk} < x)\} \to 0$, $n \to \infty, x > q$. Here we are interested in processes rather than individual r.v.'s. Thus, a sequence of p.p.'s $\{(t_{nk}, \mathbf{X}_{nk}), k \geq 1\}$, $n \geq 1$ instead of a triangular array is considered. For fixed n the r.v.'s $\mathbf{X}_{n1}, \mathbf{X}_{n2}, \ldots$ in $[0, \infty)^d$ are independent. The time points t_{nk} are chosen so that

$$0 \leq t_{n1} < t_{n2} < \ldots < t_{nk} \to \infty,\ k \to \infty, \quad t_{nk} - t_{n,k-1} \to 0,\ n \to \infty \tag{4.1}$$

Hence the counting function $k_n(t) = \max\{k : t_{nk} \leq t\}$ is finite for every fixed n and t and tends to infinity as $n \to \infty$. Now with each row we associate an extremal process $\mathbf{Y}_n$. If $\mathbf{X}_{nk}$ has lower endpoint q_{nk}, the lower curve $\mathbf{C}_n$ of $\mathbf{Y}_n$ is just $\mathbf{C}_n(t) = \vee_{k=1}^{k_n(t)} q_{nk}$, so $\mathbf{Y}_n(t) = \mathbf{C}_n(t) \vee \{\vee_{k=1}^{k_n(t)} \mathbf{X}_{nk}\}$.

We are interested in the asymptotic behavior of $\mathbf{Y}_n$ for $n \to \infty$. The points $(t_{nk}, \mathbf{X}_{nk})$ which belong to $[\mathbf{0}, \mathbf{C}_n]$ a.s. do not contribute to the limit, but only $(t_{nk}, \mathbf{C}_n(t_{nk}) \vee \mathbf{X}_{nk})$. Thus, we may start our study with a given sequence of B.p.p.

$\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ on $\mathcal{S}_n = [\mathbf{0}, \mathbf{C}_n]^c$ with m.m. μ satisfying

$$\mu_n([0,t] \times [\mathbf{0},\mathbf{x})^c) = \sum_{k=1}^{k_n(t)} \mathbf{P}(\mathbf{X}_{nk} \in [\mathbf{0},\mathbf{x})^c) < \infty \text{ for } \mathbf{x} > \mathbf{C}_n(t),\ t \geq 0. \tag{4.2}$$

Next we assume the weak convergence

$$\mathbf{Y}_n(\cdot) = \mathbf{C}_n(\cdot) \vee \{\vee_{k=1}^{k_n(\cdot)} \mathbf{X}_{nk}\} \Rightarrow \mathbf{Y}(\cdot) \tag{4.3}$$

to an extremal process $\mathbf{Y}$ with lower curve $\mathbf{C}$. Moreover, we suppose that the p.p.'s $\mathcal{N}_n$ satisfy the following "asymptotic negligibility" condition

$$\max_{\{k:t_{nk}\leq t\}} \{1 - \mathbf{P}(\mathbf{X}_{nk} < \mathbf{x})\} \to 0,\ n \to \infty,\ \mathbf{x} > \mathbf{C}(t) \tag{AN}$$

uniformly in t. Here "negligible" stands for the influence of any individual r.v. $\mathbf{X}_{nk}$ to the asymptotic behaviour of $\mathbf{Y}_n$ rather than for its size.

Remark 5. The (AN) condition on the set $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ is equivalent to condition (ii) in Theorem 1. Indeed, for $t \geq 0$ and $\mathbf{x} > \mathbf{C}(t)$ the mean measure μ_n of $\mathcal{N}_n$ satisfies $\sup_{s\leq t} \mu_n(\{s\} \times [\mathbf{0},\mathbf{x})^c) = \sup_{s\leq t} \mathbf{E}\mathcal{N}_n(\{s\} \times [\mathbf{0},\mathbf{x})^c) = \sup_{k:t_{nk}\leq t} \mathbf{P}(\mathbf{X}_{nk} \in [\mathbf{0},\mathbf{x})^c)$.

Remark 6. Convergence (4.3) implies (LC) condition.

Definition 1. *We refer to a sequence of p.p.'s $\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}), k \geq 1\}, n \geq 1$ as uniformly negligible triangular array (u.n.t.a.) if its time points t_{nk} satisfy (4.1) and its space points $\mathbf{X}_{nk}$ are row-wise independent r.v.'s satisfying the (AN) condition for some increasing right-continuous curve $\mathbf{C}(t)$.*

"Triangular" here stands to remind that $k_n(t) < \infty,\ t \geq 0$ and that only $\mathbf{X}_{n1}, \ldots, \mathbf{X}_{n,k_n(t)}$ are used to construct $\mathbf{Y}_n(t)$.

Let μ be a Radon measure and let $M(t,\mathbf{x}) := \mu([0,t] \times [0,\mathbf{x})^c)$ be its distribution function. We denote by $\mathcal{R}_c$ the set of all Radon measures on $[\mathbf{0}, \mathbf{C}]^c$ such that $e^{-M(t,\mathbf{x})}$ is df of an extremal process, i.e.
(a) $M(t,\mathbf{x})$ is right-continuous in t and left-continuous in $\mathbf{x}$. That $M(t,\mathbf{x})$ increases in t and decreases in $\mathbf{x}$ is satisfied by definition;
(b) $M(t,\mathbf{x}) < \infty$ for $\mathbf{x} \in (\mathbf{C}(t), \infty]$. This condition is satisfied by any Radon measure on $\mathcal{S}$ since for $\mathbf{x} > \mathbf{C}(t)$ the compactified set $A_{t,\mathbf{x}} = [0,t] \times [\mathbf{0},\mathbf{x})^c$ belongs to $\mathcal{S}$;
(c) $M(t,\mathbf{x}) \to 0$ for fixed $t > 0$ and $\mathbf{x} \to \infty$;
(d) The difference $M(t,\mathbf{x}) - M(s,\mathbf{x}) = \mu((s,t] \times [\mathbf{0},\mathbf{x})^c)$ for $s < t$ satisfies conditions (a) - (c) for any fixed $s \geq 0$, which hold naturally.

The following theorem characterizes the limit extremal process $\mathbf{Y}$ in (4.3) as max-id.

Theorem 4. *Let $\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ be u.n.t.a. and let $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ be an extremal process with df f and lower curve $\mathbf{C}$. Then the following statements are equivalent*
(i) $\mathbf{Y}_n \Rightarrow \mathbf{Y}$
(ii) Condition (LC) is met and

$$\sum_{k=1}^{k_n(t)} \mathbf{P}(\mathbf{X}_{nk} \in [\mathbf{0},\mathbf{x})^c) \to \mu([0,t] \times [\mathbf{0},\mathbf{x})^c), n \to \infty \tag{4.4}$$

for some measure $\mu \in \mathcal{R}_c$ and all continuity points $(t,\mathbf{x})$ of f, such that $\mathbf{x} \in \{F_t > 0\}$. Furthermore, any one of these statements is equivalent to the vague convergence of $\mathcal{N}_n$

on $\mathcal{S}$ to a Poisson point process $\mathcal{N}$ with mean measure μ and $\mathbf{Y}$ is just the associated with $\mathcal{N}$ max-id extremal process.

Proof. $(ii) \Rightarrow (i)$
In view of Theorem 1 and (4.2), both conditions (4.4) and (AN) imply $\mathcal{N}_n \stackrel{v}{\to} \mathcal{N}$, where $\mathcal{N}$ is Poisson p.p. with m.m. μ. Hence the associated with $\mathcal{N}$ extremal process $\mathbf{Y}$ is max-id and $f(t, \mathbf{x}) = \exp\{-\mu([0,t] \times [\mathbf{0}, \mathbf{x})^c)\}$. Now the (LC) condition enables to apply Theorem 2 and obtain (i).
$(i) \Rightarrow (ii)$
Conversely, we have still to show that (i) entails (4.4). Let f_n and $\mathbf{C}_n$ be df and lower curve of $\mathbf{Y}_n$. Then (i) says that for all continuity points of f and for $n \to \infty$ $f_n(t, \mathbf{x}) \to f(t, \mathbf{x})$. More precisely

$$f_n(t, \mathbf{x}) = \mathbf{P}(\mathbf{Y}_n(t) < \mathbf{x}) \sim \exp\{-\sum_{k=1}^{k_n(t)} \mathbf{P}(\mathbf{X}_{nk} \in [\mathbf{0}, \mathbf{x})^c)\} \to \begin{cases} f(t, \mathbf{x}), & \mathbf{x} \in \{F_t > 0\} \\ 0, & \text{otherwise} . \end{cases}$$

Here the sign $\sim$ is legally used since $\mathcal{N}_n, n \geq 1$ form u.n.t.a.
Now determine a measure μ on $\mathcal{S}$ by setting

$$\mu([0,t] \times [\mathbf{0}, \mathbf{x})^c) = \begin{cases} -\log f(t, \mathbf{x}), & \mathbf{x} \in \{F_t > 0\} \\ \infty, & \text{otherwise} . \end{cases}$$

This measure belongs to $\mathcal{R}_c$ and so (4.4) is met. □

With the u.n.t.a. $\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ on $\mathcal{S}_n$ given above we may also associate sum processes on $\mathcal{S}_n$ for all n, as follows

$$\mathbf{S}_n(t) = \sum_{k=1}^{k_n(t)} \mathbf{X}_{nk}$$

We are interested in necessary and sufficient conditions for the weak convergence $\mathbf{S}_n \Rightarrow \mathbf{S}$. The following theorem gives such conditions.

Theorem 5. *(Functional Extremal Criterion)*
Assume $\mathbf{C} : [0, \infty) \to [0, \infty)^d, \mathbf{C}(0) = \mathbf{0}$ is an increasing right-continuous curve and define $T = \sup\{t \geq 0 : |\mathbf{C}(t)| = 0\}$. Let $\mathcal{N}_n = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}$, on $\mathcal{S}_n$, $n \geq 1$ form u.n.t.a. and let $\mathbf{S} : [0, \infty) \to [0, \infty)^d$ be a stochastically continuous sum process on $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ with characteristic function $\psi \sim (\mathbf{a}_T, \mu)$. Then

$$\mathbf{S}_n(\cdot) = \sum_{k=1}^{k_n(\cdot)} \mathbf{X}_{nk} \Rightarrow \mathbf{S}(\cdot) \tag{4.5}$$

if and only if
(i) the associated extremal processes $\mathbf{Y}_n$ converge weakly to a stochastically continuous extremal process $\mathbf{Y}$ with lower curve $\mathbf{C}$ and df $f(t, \mathbf{x}) = e^{-\mu(A_{t,\mathbf{x}})}, t \geq 0,\ \mathbf{x} > \mathbf{C}(t)$;
(ii) the following condition holds for $t \leq T$ and $h > 0$

$$\sum_{k=1}^{k_n(t)} \mathbf{E}\left(\mathbf{X}_{nk} \mathbb{I}_{\{|\mathbf{X}_{nk}| \leq h\}}\right) \to \mathbf{a}_T(t) + \int_0^t \int_{\{|\mathbf{x}| \leq h\}} \mathbf{x} \mu(ds, d\mathbf{x}) < \infty. \tag{4.6}$$

Proof. Sufficiency:
Suppose that $\mathbf{Y}_n \Rightarrow \mathbf{Y}$ with df $f(t,\mathbf{x}) = e^{-\mu(A_{t,\mathbf{x}})}, t \geq 0, \mathbf{x} > \mathbf{C}(t)$. Hence (4.4) holds. By assumption μ is the Levy measure of the sum process $\mathbf{S}$ on $\mathcal{S}$. Thus, $\mu \in \mathcal{R}_c$ and satisfies conditions (I) - (IV). By Theorem 3 we know that μ is also a mean measure of the generating Poisson p.p. $\mathcal{N}$. On the other hand in view of Theorem 4 the sequence $\mathcal{N}_n$ converges weakly on $\mathcal{S}$ to a Poisson p.p., say $\mathcal{N}^*$, with the same mean measure μ as $\mathcal{N}$. Hence $\mathcal{N}^*$ coincides in distribution with $\mathcal{N}$ and we may consider $\mathbf{S}$ and $\mathbf{Y}$ generated by the same Poisson p.p. $\mathcal{N}$.

Since $\mathbf{S}_n$ and $\mathbf{S}$ are increasing processes it is enough to show the convergence $\mathbf{S}_n(t) \xrightarrow{d} \mathbf{S}(t)$, for all $t > 0$. Consider the case $t \leq T$. In this case $\mathbf{C} \equiv \mathbf{0}$ and the r.v.'s $\mathbf{X}_{n1}, \ldots, \mathbf{X}_{n,k_n(t)}$ for $n \geq 1$ form a null array since (AN) condition says $\mathbf{X}_{nk} \xrightarrow{\mathbf{P}} \mathbf{0}, n \to \infty$ uniformly in k. Then the convergence of the r.v. $\mathbf{S}_n(t) = \sum_{k=1}^{k_n(t)} \mathbf{X}_{nk}$ to the id r.v. $\mathbf{S}(t)$ is reduced to the classical framework of null arrays with row-wise independent r.v.'s., where conditions (4.4) and (4.6) are necessary and sufficient for the convergence $\mathbf{S}_n(t) \xrightarrow{d} \mathbf{S}(t),\ t > 0$ (cf Theorem 13.28 in Kallenberg (1997)).
Now fix $t > T$. In that case the r.v. $\mathbf{S}_n(t)$ can be represented as $\mathbf{S}_n(t) = \mathbf{S}_n(T) + \mathbf{S}_n^{(2)}(t)$ where $\mathbf{S}_n^{(2)}(t) = \sum_{T < t_{nk} \leq t} \mathbf{X}_{nk}$. Further, $\mathbf{S}_n(T) \xrightarrow{d} \mathbf{S}(T)$, so it is enough to show that $\mathbf{S}_n^{(2)}(t) \xrightarrow{d} \mathbf{S}^{(2)}(t)$, where $\mathbf{S}^{(2)}(t) = \sum_{T < T_k \leq t} \mathbf{X}_k$.
The characteristic function of $\mathbf{S}_n^{(2)}(t)$ is

$$\mathbf{E}\exp\left(i\mathbf{z}.\mathbf{S}_n^{(2)}(t)\right) = \mathbf{E}\exp\left(i\mathbf{z}.\sum_{\{k: T < t_{nk} \leq t\}} \mathbf{X}_{nk}\right) =$$

$$= \prod_{\{k: T < t_{nk} \leq t\}} \mathbf{E}\exp\left(i\mathbf{z}.\mathbf{X}_{nk}\right) = \prod_{\{k: T < t_{nk} \leq t\}} \left[\int_{[0,\mathbf{C}_n(t_{nk})]^c} e^{i\mathbf{z}.\mathbf{x}} \mathbf{P}(\mathbf{X}_{nk} \in \mathbf{dx})\right]$$

$$= \prod_{\{k: T < t_{nk} \leq t\}} \left[1 + \int_{[0,\mathbf{C}_n(t_{nk})]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mathbf{P}(\mathbf{X}_{nk} \in \mathbf{dx})\right]$$

$$= \exp \sum_{\{k: T < t_{nk} \leq t\}} \log\left[1 + \int_{[0,\mathbf{C}_n(t_{nk})]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mathbf{P}(\mathbf{X}_{nk} \in \mathbf{dx})\right]$$

Condition (LC) combined with condition (AN) imply

$$\exp \sum_{\{k: T < t_{nk} \leq t\}} \log\left[1 + \int_{[0,\mathbf{C}_n(t_{nk})]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mathbf{P}(\mathbf{X}_{nk} \in \mathbf{dx})\right] \sim$$

$$\exp \sum_{\{k: T < t_{nk} \leq t\}} \int_{[0,\mathbf{C}(t_{nk})]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mathbf{P}(\mathbf{X}_{nk} \in \mathbf{dx}) =$$

$$\exp \sum_{\{k: T < t_{nk} \leq t\}} \int_T^t \int_{[0,\mathbf{C}(s)]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mathbf{P}((t_{nk}, \mathbf{X}_{nk}) \in ds \times \mathbf{dx})$$

Finally, we get

$$\mathbf{E}\exp\left(i\mathbf{z}.\mathbf{S}_n^{(2)}(t)\right) = \exp\left[\int_T^t \int_{[0,\mathbf{C}(s)]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1) \sum_{\{k: T < t_{nk} \leq t\}} \mathbf{P}((t_{nk}, \mathbf{X}_{nk}) \in ds \times \mathbf{dx})\right]$$

$$\longrightarrow \exp\left[\int_T^t \int_{[0,\mathbf{C}(s)]^c} (e^{i\mathbf{z}.\mathbf{x}} - 1)\mu(ds, \mathbf{dx})\right] = \mathbf{E}\exp\left(i\mathbf{z}.\mathbf{S}^{(2)}(t)\right).$$

Consequently, $\mathbf{S}_n^{(2)}(t) \xrightarrow{d} \mathbf{S}^{(2)}(t)$ for $t > T$ and we conclude that $\mathbf{S}_n \Rightarrow \mathbf{S}$ on $\mathcal{S}$.
The necessity of (i) and (ii) for the convergence (4.5) is easily checked and we omit the proof.

□

5 Subordination

Here we generalize the results of Section 4 assuming that the time points of the generating B.p.p $\mathcal{N}_n$ are random. Theorems 6, 7 and 8 below are improved versions of Theorems 3, 4 and 5 in Pancheva, Mitov and Volkovich (2006).

Let us consider a sequence of extremal processes $\mathbf{Y}_n$,

$$\mathbf{Y}_n(t) = \mathbf{C}_n(t) \vee \{\vee \mathbf{X}_{nk} : T_{nk} \le t\}$$

with lower curve $\mathbf{C}_n$ and generating B.p.p. $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}) : k \ge 1\}$, defined on the open set $\mathcal{S}_n = [\mathbf{0}, \mathbf{C}_n]^c$ in $Z, n \ge 1$, where

(a) *the sequences* $\{T_{nk} : k \ge 1\}$ *and* $\{\mathbf{X}_{nk} : k \ge 1\}$ *are independent for every* $n \ge 1$ *and defined on the same probability space;*

(b) *the random time points* $\{T_{nk} : k \ge 1\}$ *are strictly increasing to infinity, i.e.* $0 \le T_{n1} < T_{n2} < \dots$;

(c) *the state points* $\{\mathbf{X}_{nk} : k \ge 1\}$ *are row-wise independent r.v.'s in* $[0, \infty)^d$.

With $\mathcal{N}_n$ we associate the counting process

$$N_n(t) = \max\{k : T_{nk} \le t\}$$

and the sum process

$$\mathbf{S}_n(t) = \sum_{k=1}^{N_n(t)} \mathbf{X}_{nk}$$

In this section we ask for relationships between the asymptotic behaviour of $\mathcal{N}_n, \mathbf{Y}_n$ and $\mathbf{S}_n$ for $n \to \infty$. To this end we impose our **basic assumption**: For every $n \ge 1$ there exists a deterministic counting function $k_n(t)$ and a random time change $\theta_n(t)$ such that

$$N_n(t) = k_n(\theta_n(t)) \quad \text{a.s.} \quad \text{for all } t > 0 \tag{BA}$$

Recall: A random time change $\theta : (0, \infty) \to (0, \infty), \theta(0) = 0$ and $\theta(s) \to \infty$ as $s \to \infty$, is stochastically continuous and has strictly increasing sample paths.

Condition (b) implies $N_n(t) < \infty$ a.s. for each n and t. Thus, $k_n(t)$ is finite and determines uniquely an associated sequence of deterministic distinct time points $0 \le t_{n1} < t_{n2} < \dots$ such that $k_n(t) = \max\{k : t_{nk} \le t\}$. Given both counting process N_n and k_n the random time change θ_n is uniquely determined at $t_{n1}, t_{n2}, \dots$ and can be defined piecewise linearly between them (see Pancheva and Jordanova (2004b)). In our model $k_n(t)$ is not arbitrary, but just a counting function that guarantees the convergence (4.4). Observe that $k_n(t)$ is not uniquely determined by (4.4) and depends on the tails $1 - \mathbf{P}(\mathbf{X}_{nk} < \mathbf{x}),\ \mathbf{x} > \mathbf{C}(t)$.

Definition 2. *The point process* $\mathcal{N}_n^{(a)} = \{(t_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ *whose state components are the same as those of* $\mathcal{N}_n$ *and whose time components are related to the time components of* $\mathcal{N}_n$ *by (BA), we term* ***accompanying point process****.*

Analogously we call the extremal process $\mathbf{Y}_n^{(a)}$ and the sum process $\mathbf{S}_n^{(a)}$ generated by $\mathcal{N}_n^{(a)}$ *accompanying extremal* - and *accompanying sum* processes to the corresponding processes $\mathbf{Y}_n$ and $\mathbf{S}_n$ generated by $\mathcal{N}_n$. We observe that $\mathbf{S}_n(t) = \sum_{k=1}^{N_n(t)} \mathbf{X}_{nk} = \sum_{k=1}^{k_n(\theta_n(t))} \mathbf{X}_{nk} = \mathbf{S}_n^{(a)}(\theta_n(t))$ and analogously $\mathbf{Y}_n = \mathbf{Y}_n^{(a)} \circ \theta_n$.

Now, let $\mathbf{C} : [0, \infty) \to [0, \infty)^d$ be an increasing right-continuous curve and put $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$. From the previous Sections 2 and 4 we already know that conditions (4.4), (LC), (AN) and (4.6) determine uniquely the relationship between the accompanying processes. The question is what additional condition we need in order to claim the convergence of the new processes $\mathcal{N}_n, \mathbf{Y}_n$ and $\mathbf{S}_n$.

Theorem 6. *Let* $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}) : k \geq 1\}$ *be B.p.p on* $\mathcal{S}_n$ *satisfying conditions (a)-(c) whose counting processes obey the basic assumption (BA). Suppose the random time changes* θ_n *are weakly convergent to a random time change* Λ*. If the sequence of the accompanying p.p.'s* $\mathcal{N}_n^{(a)}$ *is vaguely convergent on* $\mathcal{S}$ *to a simple in time Poisson p.p.* $\mathcal{N}$ *with mean measure* μ*, then the sequence* $\mathcal{N}_n$ *is weakly convergent to a Cox p.p.* $\tilde{\mathcal{N}}$ *with mean measure* $\tilde{\mu}$ *defined on an open subset* $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ *and satisfying*

$$\tilde{\mu}(A_{t,\mathbf{x}}) = \mathbf{E}\mu([0, \Lambda(t)] \times [\mathbf{0}, \mathbf{x})^c)$$

for all points $(t, \mathbf{x})$ *such that* $A_{t,\mathbf{x}}$ *belongs to* $\tilde{\mathcal{S}}$*.*

Theorem 7. *Let* $\mathbf{Y}_n$ *be extremal processes on* $\mathcal{S}_n$ *generated by the point processes* $\mathcal{N}_n$ *from Theorem 6,* $n \geq 1$*. Suppose that conditions (AN), (LC) and (4.4) hold. If* θ_n *converges weakly to a random time change* Λ*, then the sequence* $\mathbf{Y}_n$ *is weakly convergent to the composition* $\tilde{\mathbf{Y}} := \mathbf{Y} \circ \Lambda$ *with df* $\tilde{f}$ *where* $\mathbf{Y}$ *is a max-id extremal process and*

$$\tilde{f}(t, \mathbf{x}) = \mathbf{E}e^{-\mu([0,\Lambda(t)]\times[\mathbf{0},\mathbf{x})^c)}.$$

Theorem 8. *Let* $\mathbf{S}_n$ *be sum process on* $\mathcal{S}_n$ *generated be the point process* $\mathcal{N}_n$ *from Theorem 6,* $n \geq 1$*. Suppose*
(i) $\mathbf{Y}_n^{(a)} \Rightarrow \mathbf{Y}$*, a stochastically continuous extremal process with df* $f(t, \mathbf{x}) = e^{-\mu(A_{t,\mathbf{x}})}$, $\mathbf{x} > \mathbf{C}(t),\ t \geq 0$;
(ii) $\sum_{k=1}^{k_n(t)} \mathbf{E}\left(\mathbf{X}_{nk}\mathbb{I}\{|\mathbf{X}_{nk}| \leq h\}\right) \to \mathbf{a}_T(t) + \int_0^t \int_{\{|\mathbf{x}|\leq h\}} \mathbf{x}\mu(ds, d\mathbf{x}) < \infty$ *for* $t \leq T$*, and* $h > 0$;
(iii) $\theta_n \Rightarrow \Lambda$*, a random time change.*
Then $\mathbf{S}_n \Rightarrow \tilde{\mathbf{S}} = \mathbf{S} \circ \Lambda$ *where* $\mathbf{S}$ *is a stochastically continuous sum process with characteristic function* $\psi \sim (\mathbf{a}_T, \mu)$ *and*

$$\tilde{\psi}_t(\mathbf{z}) = \mathbf{E}e^{i\mathbf{z}.\tilde{\mathbf{S}}(t)} = \mathbf{E}\psi_{\Lambda(t)}(\mathbf{z})$$

The proofs are direct consequences of our main results in Sections 2 and 4 and the continuity of composition theorem (cf Th. 13.2.3 in Whitt (2002)).

6 Conclusions

Relationship between sum and extremal processes over explosion area generated by a Poisson point process have been obtained. These sum and extremal processes arise as a weak limits for normalized sums and maxima of independent random vectors. The limit processes can be used as approximations for real processes in insurance (total and extremal claim amounts) and in operational risk modeling. The results are successfully applied for approximating aggregate and extremal operational losses in a forthcoming paper under the assumption that the loss amounts $X_k, k = 1, 2, 3, \ldots$ follow Pareto distribution H_k.

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