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Nonlinear Normalization in Limit Theorems for Extremes

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Abstract It is well known that under linear normalization a sequence of maxima of iid random variables converges in distribution to one of the three max-stable laws: Frećhet, Gumbel, and Weibul. During the last two decades E. Pancheva and her collaborators developed the limit theory for extremes and extremal processes under non-linear but monotone increasing normalizing mappings. In the present we discuss the general form of the limit law for the maxima of iid random variables under monotone normalization. A theorem for the rate of convergence is also proved.

Keywords extreme value · nonlinear normalization · limit theorems · domain of attraction

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1 Introduction

Any limit theorem for convergence of normalized maxima of iid random variables to a max-stable law G separates a subclass of distribution functions (d.f.'s) $MDA(G)$ called max-domain of attraction of G. Thus, if we use a wider class of normalizing mappings than the linear ones, we get a wider class of limit laws which can be used in solving approximation problems. Another reason for using nonlinear normalization concerns the problem of refining the accuracy of approximation in the limit theorems: by using relatively non difficult monotone mappings in certain cases we can achieve a better rate of convergence. In the last two decades E. Pancheva and her collaborators were investigating various limit theorems for extremes and extremal processes using

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as normalizing mappings the so-called max - automorphisms (see e.g. Pancheva 2010): continuous and strictly increasing in each coordinates. The max-automorphisms preserve the max-operation, i.e. $L(X \vee Y) = L(X) \vee L(Y)$, there exist the inverse mappings L^{-1} and they form a group w.r.t. the composition. We denote it by GMA.

Choosing mappings from GMA for normalization in the limit theorems, we are imposed to change the notion of type (F) for a non degenerate d.f. F: We say a d.f. G belongs to $type(F)$ if there exists $T \in GMA$ such that $G = F \circ T$. Now the three extreme value distributions (Fréchet's Φ_α, Gumbel's Λ, and Weibull's Ψ_α) belong to the same type, the max-stable type $(\mathcal{MS})$.

Let us recall the major notions of the univariate max-model: A nondegenerate d.f. G is called max-stable if there exists a continuous one-parameter group (c.o.g.) $\mathcal{L} = \{\mathbb{L}_t : t > 0\}$ in GMA such that for all $t > 0$

$$G^t(x) = G(\mathbb{L}_t(x)). \tag{1}$$

A d.f. G satisfying (1) has to be continuous and strictly increasing. Conversely, given G, let us consider (1) as a functional equation for the unknown $\mathcal{L}$. Solving it one obtains that there exists a continuous and strictly increasing mapping

$$h : Support(G) \leftrightarrow (-\infty, \infty)$$

such that

$$\mathbb{L}_t(x) = h^{-1}(h(x) - \log t), \quad t > 0. \tag{2}$$

Now (1) is equivalent to

$$G(x) = \exp\{-e^{-h(x)}\}, \tag{3}$$

and this is the general explicit form of any max-stable d.f. We denote for any d.f. G,

$$l_G = \inf\{x : G(x) > 0\}, \quad r_G = \sup\{x : G(x) < 1\}.$$

Remark 1 Note that representation (3) can be expressed also in the form

$$G(x) = \exp\{-e^{-c_1 h_1(x)}\}, \quad c_1 > 0 \tag{3'}$$

or in the form

$$G(x) = \exp\{-c_2 e^{-h_2(x)}\}, \quad c_2 > 0 \tag{3''}$$

for aiming a parametrization of the class $\mathcal{MS}$. Under this parametrization the c.o.g. $\mathcal{L}$ remains the same: since $h(x) = c_1 h_1(x) = h_2(x) - \log c_2$ then

$$\mathbb{L}_t(x) = h^{-1}(h(x) - \log t) = h_1^{-1}(h_1(x) - \frac{1}{c_1}\log t) = h_2^{-1}(h_2(x) - \log t).$$

In this connection, the claims in (Sreehari 2009) are unfounded.

The convergence to type theorem (CTT) is the main tool for proving limit theorems for cumulative extremes. A convergence to type takes place if both convergences $F_n \xrightarrow{w} F$ and $F_n \circ T_n \xrightarrow{w} G$, where $T_n \in GMA$, imply $G \in type(F)$, i.e. there exists a $T \in GMA$ such that $G = F \circ T$. Using here max-automorphisms we are confronted with similar difficulties as if we were working in a space with infinite dimensions. Let $\mathcal{R}_f$ be the set of all sequences $\{T_n\} \subset GMA$ satisfying the conditions

a) $T_n(x) \geq x$,

b) $h \leq T_n(x+h) - T_n(x) \leq f(h) \to 0, \quad h \to 0,$

where $f : \mathbf{R} \to \mathbf{R}$ and $h \in (0,1)$. Denote $\mathcal{R} = \bigcup_f \mathcal{R}_f$. The sequences $\{T_n\}$ from $\mathcal{R}$ are equicontinuous and bounded from below. If in addition there exists a limit mapping T, then the right hand side of b) gives the continuity of T and the left hand side of b) supplies its strong monotony, i.e. $T \in GMA$.

Now, the CTT in our model claims: *The compactness (w.r.t. the pointwise convergence) of the normalizing sequence T_n is necessary and sufficient for a convergence to type.* Unfortunately, this new formulation of CTT makes it difficult for application. This is the reason for RESTRICTING our investigation to REGULAR normalizing sequences only. In this way we lose in generality but win in clarity.

Definition 1 We refer to a sequence $\{L_n\} \subset GMA$ as regular for $n \to \infty$ on a set $S \times T$ if for every $x \in S$ and $t \in T$ there exists a limiting max-automorphism

$$\mathbb{L}_t(x) = \lim_{n \to \infty} L^{-1}_{[nt]} \circ L_n(x) \tag{4}$$

uniformly on compact subsets of T and the mapping $t \to \mathbb{L}_t$ is one-to-one.

The main advantage of the restriction to the regular normalizing sequences is that instead of using CTT we use the continuity of the composition.

Let $X_1, X_2, \ldots$ be iid r.v.s with d.f. F. Let G be a nondegenerate d.f. Suppose that there exists a regular on $(l_G, r_G) \times (0, \infty)$ normalizing sequence $\{L_n\}$ such that

$$F^n(L_n(x)) \xrightarrow{w} G(x). \tag{5}$$

Using the regularity of the sequence we see immediately that the limit law G satisfies functional equation (1), hence G is max-stable w.r.t. the c.o.g. $\mathcal{L} = \{\mathbb{L}_t, t > 0\}$ determined by (4). If (5) is met we say that F belongs to the max-domain of attraction of G w.r.t.$\mathcal{L}$, briefly $F \in MDA(G)$.

We underline that the regularity of the normalizing sequence is requested on (l_G, r_G) with the following example.

Example 1 Let $F(x) = \Lambda(x)$ and $L_n(x) = \begin{cases} e^n \left(x - \frac{1}{n}\right) & if \;\; x \leq \frac{1}{n} \\ \log nx & if \;\; x > \frac{1}{n}. \end{cases}$ Then (5) is met with $G(x) = \Phi_1(x)$ and

$$\lim_{n \to \infty} L^{-1}_{[nt]} \circ L_n(x) = \mathbb{L}_t(x)$$

where

$$\mathbb{L}_t(x) = \begin{cases} \begin{cases} \frac{x}{t}, & x > 0, \\ -\infty, & x \leq 0 \end{cases} & \text{for} \quad t \in (0,1) \\ \begin{cases} \frac{x}{t}, & x > 0, \\ 0, & x \leq 0 \end{cases} & \text{for} \quad t \geq 1. \end{cases}$$

Hence $\{L_n\}$ is regular on $(0, \infty) \times (0, \infty)$, Φ_1 is max-stable with $h(x) = \log x$, $\mathbb{L}_t(x) = \exp(\log x - \log t)$.

Theorem 5 in (Pancheva 1984) says: *A nondegenerate d.f. F belongs to $MDA(G)$ iff*

$$1 - F(x) = [1 + o(1)]R(h(x))e^{-h(x)}, \quad x \to r_F,$$

where $R(x)$ is a regularly varying function at infinity. The normalizing mappings can be chosen as

$$L_n(x) = h^{-1}\{h(x) + \log[nL(\log n)]\}.$$

Sreehari [7] pointed out that the necessary part of the above statement is wrong and proposed the following theorem: *If a nondegenerate d.f. $F \in MDA(G)$ then there exists a sequence of positive functions $\{L^*(x; n)\}$ such that*

$$\frac{K\{h(x) + \log(nL^*(x; n))\}}{L^*(x; n)} \to 1, \ \text{as } n \to \infty, \ \text{for } x \in (l_F, r_F), \tag{6}$$

where $K(x) = [1 - F \circ h^{-1}(x)]e^x$. Conversely, if for some strictly increasing continuous function $h(x)$ and a sequence of positive functions $\{L^*(x; n)\}$ equation (6) holds then $F \in MDA(G), \ G(x) = e^{-e^{-h(x)}}$. In this case $L_n(x)$ can be chosen as

$$L_n(x) = h^{-1}\{h(x) + \log[nL^*(x; n)]\}. \tag{7}$$

We are deeply thankful to Professor Sreehari for discovering the boring mistake. Yet, in the framework of our max-model the suggested normalization (7) cannot be adopted: the variables x and n in L^* are not separated and in general one can not check if (7) defines (or does not define) a regular normalizing sequence. The aim of the present paper is to give a revised answer to the problem of conditions for $F \in MDA(G)$ and to show the advantage of using regular mappings for normalization: they may yield a better rate of convergence to the limit max-stable distributions. They may also yield non-degenerate max-stable limits for distributions like the Poisson (which were previously thought not to have non-degenerate limits). We start with several illustrative examples, then in Section 3 we state and prove our main results.

2 Examples

Example 2 Let $X_1, X_2, \ldots$ be i.i.d. r.v. with c.d.f. $F(x) = 1 - x^{-x}, \ x \geq 1$. Denote by $M_n = \max\{X_1, X_2, \ldots, X_n\}$. We want to find a normalizing sequence $L_n(x)$ such that

$$P(M_n \leq L_n(x)) = P\{L_n^{-1}(M_n) \leq x\} \to \text{ proper limit distribution.}$$

It is natural to assume that the function $U(x) := \dfrac{1}{1 - F(x)} = x^x, \quad x \geq 1$ will play an important role. Let us check some properties of $U(x)$. We have that $U'(x) = x^x(1 + \ln x) > 0$ for every $x \geq 1$. So, $U(x)$ is strictly increasing and continuous on the interval $[1, \infty)$, $U(1) = 1$, and $U(x) \uparrow \infty$ as $x \to \infty$, and

$$U : [1, \infty) \to [1, \infty).$$

Therefore, there exists the inverse function $U^{-1}(x)$, that is

$$U(U^{-1}(x)) = U^{-1}(x)^{U^{-1}(x)} = x \text{ and } U^{-1}(U(x)) = x$$

for every $x \geq 1$. The function $U^{-1}(x)$ is also strictly increasing on $[1,\infty)$, $U^{-1}(1) = 1$, $U^{-1}(x) \to \infty$ as $x \to \infty$, and

$$U^{-1} : [1,\infty) \to [1,\infty).$$

2.1. Fréchet limit distribution. Let us denote $L_n(x) = U^{-1}(nx)$ and then $L_n^{-1}(x) = \frac{U(x)}{n}$, for every $x > 0$ and $n = 1, 2, \ldots$

We prove that as $n \to \infty$,

$$P\left\{L_n^{-1}(M_n) \leq x\right\} = P\left\{\frac{U(M_n)}{n} \leq x\right\} \quad \to \quad \exp(-1/x), \qquad x > 0.$$

Indeed

$$\begin{aligned}
&P\left\{\frac{U(M_n)}{n} \leq x\right\} = P\left\{M_n \leq U^{-1}(nx)\right\} = \left(P\left\{X_1 \leq U^{-1}(nx)\right\}\right)^n \\
&= \left(1 - U^{-1}(nx)^{-U^{-1}(nx)}\right)^n = \left(1 - \frac{1}{U^{-1}(nx)^{U^{-1}(nx)}}\right)^n \\
&= \left(1 - \frac{1}{nx}\right)^n \to \exp(-1/x) = \Phi_1(x), \quad n \to \infty.
\end{aligned}$$

The sequence $L_n(x) = U^{-1}(nx)$ is regular. For this one has only to check that for $t > 0$, $L_{[nt]}^{-1} \circ L_n(x) \to \mathbb{L}_t(x) = x/t$. Recall, $\Phi_1(x)$ is max-stable d.f. w.r.t. the c.o.g. $\{\mathbb{L}_t(x) = \frac{x}{t},\ t \geq 0\}$. Indeed

$$L_{[nt]}^{-1} \circ L_n(x) = \frac{U(L_n(x))}{[nt]} = \frac{U(U^{-1}(nx))}{[nt]} = \frac{nx}{[nt]} \to \mathbb{L}_t(x) = x/t,\ n \to \infty.$$

2.2. Gumbel limit distribution. It appears that there exists another nonlinear normalization for the sequence M_n which leads to the Gumbel limit distribution. In other words we find a normalizing sequence $L_n(x)$ such that

$$P\{M_n \leq L_n(x)\} = P\{L_n^{-1}(M_n) \leq x\} \to \exp(-e^{-x}) = \Lambda(x).$$

Let us denote $L_n(x) = U^{-1}(ne^x)$ and then $L_n^{-1}(x) = \log \frac{U(x)}{n}$, for every $x > 0$ and $n = 1, 2, \ldots$. Then as $n \to \infty$,

$$P\left\{\log \frac{U(M_n)}{n} \leq x\right\} \quad \to \quad \exp(-e^{-x}), x \in (-\infty, \infty).$$

Indeed

$$\begin{aligned}
&P\left\{\log \frac{U(M_n)}{n} \leq x\right\} = P\left\{M_n \leq U^{-1}(ne^x)\right\} = \left(P\left\{X_1 \leq U^{-1}(ne^x)\right\}\right)^n \\
&= \left(1 - U^{-1}(ne^x)^{-U^{-1}(ne^x)}\right)^n = \left(1 - \frac{1}{U^{-1}(ne^x)^{U^{-1}(ne^x)}}\right)^n \\
&= \left(1 - \frac{1}{ne^x}\right)^n \to \exp(-e^{-x}), \quad n \to \infty.
\end{aligned}$$

In order to prove that the sequence L_n is regular one has to check that for $t > 0$, $L_{[nt]}^{-1} \circ L_n(x) \to \mathbb{L}_t(x) = x - \log t$. Recall, $\Lambda(x)$ is max-stable w.r.t. c.o.g. $\{\mathbb{L}_t = x - \log t,\ t \geq 0\}$, hence $h(x) = x$. Indeed,

$$L_{[nt]}^{-1} \circ L_n(x) = \log \frac{U(L_n(x))}{[nt]}$$
$$= \log \frac{U(U^{-1}(ne^x))}{[nt]} = \log \frac{ne^x}{[nt]} = x + \log \frac{n}{[nt]} \to x - \log t, \quad n \to \infty.$$

Remark 2 The nonlinear normalization $L_n(x) = U^{-1}(ne^x)$ in the Example 2.2 can not be represented in an explicit form, but $U^{-1}(.)$ can be determined asymptotically as the solution of the equation $\log x + \log\log x + t = 0$ (see e.g. (de Bruijn 1958)).

2.3. Linear normalization. Since the tail of the d.f. $F(x) = 1 - x^{-x}, \ \ x \geq 1$ is very light there should exist sequences $a_n > 0$ and b_n such that

$$P\{M_n \leq a_n x + b_n\} \to e^{-e^{-x}}, \ \ x \in (-\infty, \infty).$$

Note, the above relation is equivalent to

$$n(1 - F(a_n x + b_n)) \to e^{-x}. \tag{8}$$

The normalizing sequences can be chosen as follows $b_n = U^{-1}(n)$ and $a_n = \frac{1}{\log b_n}, n \geq 2$. For every $n \geq 2$ let us mention that $b_n^{b_n} = n$ and $b_n \log b_n = \log n$. Then

$$n(1 - F(a_n x + b_n)) = n \left(\frac{x}{\log b_n} + b_n \right)^{-\left(\frac{x}{\log b_n} + b_n \right)}$$
$$= n \left[b_n \left(\frac{x}{b_n \log b_n} + 1 \right) \right]^{-\left[b_n \left(\frac{x}{b_n \log b_n} + 1 \right) \right]}$$
$$= n \left(b_n^{-b_n} \right)^{\left(\frac{x}{\log n} + 1 \right)} \left(\frac{x}{\log n} + 1 \right)^{-(\log n)\left[\frac{b_n}{\log n} \left(\frac{x}{\log n} + 1 \right) \right]}$$
$$= n^{-\frac{x}{\log n}} \left(\frac{x}{\log n} + 1 \right)^{-(\log n)\left[\frac{b_n}{\log n} \left(\frac{x}{\log n} + 1 \right) \right]}$$
$$= e^{-x} \left(\frac{x}{\log n} + 1 \right)^{-(\log n)\left[\frac{b_n}{\log n} \left(\frac{x}{\log n} + 1 \right) \right]}.$$

We observe that

$$\frac{b_n}{\log n} = \frac{1}{\log b_n} \to 0, \quad n \to \infty$$

because $b_n \uparrow \infty$, as $n \to \infty$. Using this and the fact that

$$\left(\frac{x}{\log n} + 1 \right)^{-\log n} \to e^{-x},$$

we obtain

$$\left(\frac{x}{\log n} + 1 \right)^{-(\log n)\left[\frac{b_n}{\log n} \left(\frac{x}{\log n} + 1 \right) \right]} \to \left(e^{-x} \right)^0 = 1,$$

which complete the proof. After some standard calculations one can see that the sequence of linear transforms $L_n(x) = \frac{x}{\log b_n} + b_n$ is regular.

Example 3 Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with standard exponential d.f.

$$F(x) = \begin{cases} 1 - e^{-x}, \ x > 0, \\ 0, \qquad\quad x \le 0. \end{cases}$$

3.1. Linear normalization. It is well known that the sequences $a_n = 1$, $b_n = \log n$, $n = 1, 2, \ldots$ provide that for every fixed $x \in \mathbb{R}$,

$$(F(a_n x + b_n))^n = \left(1 - e^{-(a_n x + b_n)}\right)^n = \left(1 - e^{-(x + \log n)}\right)^n$$

$$= \left(1 - e^{-x} e^{-\log n}\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n \to e^{-e^{-x}}, \ n \to \infty.$$

The sequence $L_n(x) = x + \log n$ is regular.

3.2. Nonlinear normalization

3.2.1. Gumbel limit distribution. Let us define

$$U(x) = \frac{1}{1 - F(x)} = \begin{cases} e^x, \ x > 0, \\ 1, \ \ x \le 0 \end{cases}$$

and its inverse

$$U^{-1}(x) = \begin{cases} \log x \ \ x > 1, \\ -\infty, \ x \le 1. \end{cases}$$

Take the normalizing monotone transforms $L_n(x) = U^{-1}(ne^x)$. Assume that $x \in \mathbb{R}$ is fixed, then $ne^x > 1$ for every $n > e^{-x}$ and then $L_n(x) = \log(ne^x) = x + \log n$ which coincides with the linear normalization given above and $F^n(L_n(x)) \to \Lambda(x)$. This is not surprising because the exponential distribution belongs to the normal max-domain of attraction of Λ ($NMDA(\Lambda)$). Hence the normalizing sequence

$$L_n(x) = \mathbb{L}_{1/n}(x) = x + \log n$$

can not be other than linear (or asymptotically equivalent to a linear one).

3.2.2. Fréchet limit distribution. Recall Φ_α is max-stable w.r.t. $\mathbb{L}_t(x) = xt^{-1/\alpha} = \exp\{\frac{1}{\alpha}(\alpha \log x - \log t)\}$. Hence for $\alpha > 0$,

$$h(x) = \begin{cases} \alpha \log x, \ x > 0, \\ -\infty, \qquad x \le 0. \end{cases} \tag{9}$$

Now we take the following monotone normalizing sequence

$$L_n(x) = U^{-1}(ne^{h(x)}) = \log(nx^\alpha) = \alpha \log x + \log n \tag{10}$$

and obtain the convergence

$$(F(L_n(x)))^n = (1 - e^{-L_n(x)})^n = (1 - e^{-\log(nx^\alpha)})^n = (1 - \frac{1}{nx^\alpha})^n \to e^{-x^{-\alpha}}, \ \ n \to \infty.$$

Therefore under the regular normalizing sequence (10) the exponential distribution belongs to the $MDA(\Phi_\alpha)$.

Remark 3 Note that if using nonlinear normalizing sequences, the classical relation

$$F \in MDA(\Phi_\alpha) \Leftrightarrow 1 - F \in RV_{-\alpha}$$

is not true, as Examples 2 and 3 show.

Example 4 Let $X_1, X_2, \dots$ be i.i.d. r.v.s with Pareto distribution, i.e.

$$F(x) = \begin{cases} 1-(1+x)^{-\alpha}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

where $\alpha > 0$.

4.1. Linear normalization. It is well known that the sequences $a_n = n^{1/\alpha}$, $b_n = -1$, $n = 1, 2, \dots$ provide that for every fixed $x > 0$,

$$(F(a_n x + b_n))^n = \Big(1 - (1 + a_n x + b_n)^{-\alpha}\Big)^n = \Big(1 - (1 + n^{1/\alpha} x - 1)^{-\alpha}\Big)^n$$

$$= \Big(1 - n^{-1} x^{-\alpha}\Big)^n = \Big(1 - \frac{x^{-\alpha}}{n}\Big)^n \to e^{-x^{-\alpha}}, \ n \to \infty,$$

i.e. Pareto distribution belongs to $NMDA(\Phi_\alpha)$ with the regular normalizing sequence $L_n(x) = n^{1/\alpha} x - 1$.

4.2. Nonlinear normalization.

4.2.1. Fréchet limit distribution. Let us define

$$U(x) = \frac{1}{1 - F(x)} = \begin{cases} (1+x)^\alpha, & x > 0, \\ 1, & x \le 0 \end{cases}$$

and its inverse

$$U^{-1}(x) = \begin{cases} x^{1/\alpha} - 1 & x > 1, \\ -\infty, & x \le 1. \end{cases}$$

Take $h(x)$ as in (9) and define the monotone normalizing sequence

$$L_n(x) = U^{-1}(n e^{h(x)}) = n^{1/\alpha} x - 1.$$

It is in fact the linear transform given above.

4.2.2. Gumbel limit distribution. Put now $h(x) = x$ for $x \in \mathbb{R}$ and define the regular normalizing transforms $L_n(x) = (n e^x)^{1/\alpha} - 1$. Then we have

$$F^n(L_n(x)) = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}, \quad n \to \infty,$$

thus the Pareto d.f. belongs to $MDA(\Lambda)$ w.r.t. the above normalizing sequence.

3 Main results

Let F be an arbitrary nondegenerate d.f. Denote again $U(x) = \dfrac{1}{1 - F(x)}$. The mapping $U : (l_F, r_F) \to (1, \infty)$ is monotone increasing.

Lemma 1 *There exists a continuous and strictly increasing function $g(x)$ such that*

$$\frac{g(x)}{U(x)} \to 1, \quad \text{as } x \to r_F, \tag{11}$$

if and only if U is asymptotically continuous at r_F, i.e.

$$\frac{U(x+0)}{U(x-0)} \to 1, \quad \text{as } x \to r_F. \tag{12}$$

This statement is a light modification of Lemma 2 (Faktorovich 1989). The proof goes in the same way.

Theorem 1 *(***On max-domain of attraction***). Let $F \in MDA(H)$, $H(x) = e^{-e^{-h(x)}}$. Then F is asymptotically continuous at r_F and the normalizing sequence L_n can be taken as*

$$L_n(x) = g^{-1}(ne^{h(x)}), \tag{13}$$

where g is continuous and strictly increasing on (l_F, r_F) and satisfies (11). Conversely, let F be asymptotically continuous at r_F and let $h : (l_F, r_F) \leftrightarrow (-\infty, \infty)$ be continuous and strictly increasing. Then there exists a continuous and strictly increasing function g, such that the sequence $g^{-1}(ne^{h(x)})$ is regular and normalizes the convergence

$$F^n(L_n(x)) \to \exp\{-e^{-h(x)}\}, \quad n \to \infty, \tag{14}$$

i.e. $F \in MDA(H)$.

Remark 4 Roughly speaking, Theorem 1 says that, given F is asymptotically continuous at its right endpoint, then $F^n(L_n(x)) \to H(x)$ iff the tail of F, the tail of H and the regular normalizing sequence L_n are connected by the asymptotic relation

$$L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow} (n.e^{h(x)}), \quad n \to \infty. \tag{15}$$

Here $U^{\leftarrow}$ is left continuous inverse of U. The later is equivalent to

$$n(1 - F(L_n(x))) \to e^{-h(x)}, \quad n \to \infty.$$

Proof (of Theorem 1) Let $F \in MDA(H)$. Assume that F is not asymptotically continuous at r_F, but has mass $F(x-0) \to p$, as $x \to r_F$, $0 < p < 1$. Then

$$\frac{F(x+0)}{F(x-0)} \to \frac{1}{p} \neq 1, \quad \text{as } x \to r_F.$$

For x fixed and $n \to \infty$ the normalizing sequence $L_n(x) \uparrow r_F$, hence $F(L_n(x)) \to p$ and $F^n(L_n(x)) \sim p^n \to 0$ in contradiction to the assumption $F \in MDA(H)$. Thus F has to be asymptotically continuous at r_F. In this case, by Lemma 1, there exists a strictly increasing and continuous function g, with

$$g(x) \sim \left(\frac{1}{1-F}\right)(x) = U(x), \quad \text{as} \quad x \to \infty.$$

The inverse function $g^{-1}(x)$ exists. It is strictly increasing and $g^{-1}(x) \uparrow \infty$, as $x \to \infty$. Therefore

$$U(g^{-1}(x)) \sim g(g^{-1}(x)) \sim x, \quad \text{as } x \to \infty.$$

The sequence $L_n(x) = g^{-1}(ne^{h(x)})$ satisfies $F^n(L_n(x)) \to H(x),\ n \to \infty$. Indeed,

$$\begin{aligned}
&\mathbf{P}\left\{\bigvee_{i=1}^{n} X_i \leq L_n(x)\right\} = \mathbf{P}\left\{\bigvee_{i=1}^{n} X_i \leq g^{-1}(ne^{h(x)})\right\} \\
&= \mathbf{P}^n\left\{X_1 \leq g^{-1}(ne^{h(x)})\right\} = \left[1 - (1 - F(g^{-1}(ne^{h(x)})))\right]^n \\
&= \left[1 - \frac{1}{U(g^{-1}(ne^{h(x)}))}\right]^n \sim \left[1 - \frac{1}{ne^{h(x)}}\right]^n = \left[1 - \frac{e^{-h(x)}}{n}\right]^n \to \exp\{-e^{-h(x)}\},
\end{aligned}$$

as $n \to \infty$.

Besides, the sequence L_n is regular because for $t > 0$,

$$L_{[nt]}^{-1} \circ L_n(x) = h^{-1}\left(\log \frac{g(L_n(x))}{[nt]}\right) = h^{-1}\left(\log \frac{g\left(g^{-1}(ne^{h(x)})\right)}{[nt]}\right)$$
$$= h^{-1}\left(\log \frac{ne^{h(x)}}{[nt]}\right) = h^{-1}\left(\log e^{h(x)} + \log \frac{n}{[nt]}\right)$$
$$= h^{-1}\left(h(x) + \log \frac{n}{[nt]}\right) \to h^{-1}\left(h(x) - \log t\right) = \mathbb{L}_t(x), \quad n \to \infty.$$

Similarly one shows the converse part. □

Corollary 1 *1. Let $F \in MDA(\Phi_\alpha)$. Then $h(x) = \alpha \log x$ and $L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(nx^\alpha)$. The function $R(x) = nx^\alpha$ is regularly varying at infinity, hence $U(L_n(x)) \in RV_\alpha$. Since $U(y) = \frac{1}{1-F(y)}$ for $y \to r_F$ we conclude that $\bar{F} \circ L_n \in RV_{-\alpha}$. This means, in our max-model with nonlinear normalizing sequences the condition $\bar{F} \in RV_{-\alpha}$ is not more a necessary and sufficient condition for $F \in MDA(\Phi_\alpha)$, but $\bar{F} \circ L_n \in RV_{-\alpha}$ (cf. Ex 2, 3 and 4).*

2. Let $F \in MDA(\Lambda)$. Then $h(x) = x$ and $L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(ne^x)$. Hence

$$\frac{1}{n}\left(\frac{1}{1-F}\right)(L_n(x)) \to e^x, \qquad n \to \infty.$$

Choose $y_n \uparrow r_F$ such that $1 - F(y_n) = \frac{1}{n}$. Then for $U = \left(\frac{1}{1-F}\right)^{\leftarrow}$

$$\frac{U^{\leftarrow}(L_n(x))}{U^{\leftarrow}(y_n)} \to e^x, \quad n \to \infty.$$

The converse is also true (cf. (de Haan 1970)).

Remark 5 Examples 2, 3, and 4 from Section 2 show that a distribution may belong to MDA of two different max-stable laws. It will be very wrong to conclude that "domains of attractions of different types are not disjoint if using monotone normalization" as read in some authors. In fact, if using monotone normalization, there is only one type of max-stable laws!

Theorem 2 *Let $F \in MDA(H), H(x) = e^{-e^{-h(x)}}$, w.r.t. the regular normalizing sequence L_n, defined in Theorem 1. If $g(x) = U(x)$ then*

$$\left|F^n(L_n(x)) - H(x)\right| = O\left(1/n\right), \quad n \to \infty.$$

Remark 6 If the function $g(x)$ is asymptotically equivalent to $U(x)$ then the rate of convergence depends also on the rate of convergence in the asymptotic relation $\frac{g(x)}{U(x)} \to 1$ as $x \to \infty$.

Proof Since $F^n(L_n(x)) = \left[1 - \frac{e^{-h(x)}}{n}\right]^n$ then we have to estimate

$$\left|\left[1 - \frac{e^{-h(x)}}{n}\right]^n - e^{-e^{-h(x)}}\right|$$

The following inequality is valid for every u, v such that $0 < v < u$ and $\alpha > 1$

$$\alpha(u-v)v^{\alpha-1} < u^{\alpha} - v^{\alpha} < \alpha(u-v)u^{\alpha-1}.$$

Let $x > 0$ be fixed. Then

$$e^{-e^{-h(x)}} - \left(1 - \frac{e^{-h(x)}}{n}\right)^n = \left(e^{-e^{-h(x)}/n}\right)^n - \left(1 - \frac{e^{-h(x)}}{n}\right)^n$$
$$\leq n\left(e^{-e^{-h(x)}/n} - 1 + \frac{e^{-h(x)}}{n}\right)\left(e^{-e^{-h(x)}/n}\right)^{n-1}$$
$$e^{-e^{-h(x)}} - \left(1 - \frac{e^{-h(x)}x}{n}\right)^n = \left(e^{-e^{-h(x)}/n}\right)^n - \left(1 - \frac{e^{-h(x)}}{n}\right)^n$$
$$\geq n\left(e^{-e^{-h(x)}/n} - 1 + \frac{e^{-h(x)}}{n}\right)\left(1 - \frac{e^{-h(x)}}{n}\right)^{n-1}$$

From the power series for the exponential function we have that as $n \to \infty$,

$$\left(e^{-e^{-h(x)}/n} - 1 + \frac{e^{-h(x)}}{n}\right) = \frac{e^{-2h(x)}}{2n^2}(1 + o(1)).$$

Since

$$\left(e^{-e^{-h(x)}/n}\right)^{n-1} \to e^{-e^{-h(x)}}$$

and

$$\left(1 - \frac{e^{-h(x)}}{n}\right)^{n-1} \to e^{-e^{-h(x)}}$$

as $n \to \infty$ we can conclude that

$$\left| e^{-e^{-h(x)}} - \left(1 - \frac{e^{-h(x)}}{n}\right)^n \right| = O\left(\frac{1}{n}\right)$$

as $n \to \infty$. □

Corollary 2 *Let L_n and T_n be two normalizing sequences of max-automorphisms, such that*

$$n(1 - F(L_n(x))) \to e^{-h(x)}, \tag{16}$$
$$n(1 - F(T_n(x))) \to e^{-h(x)} \tag{17}$$

for h continuous and strictly increasing. Then both sequences are regular and asymptotically equivalent in the sense that

$$L_n^{-1}(T_n(x)) \to x, \quad n \to \infty.$$

Conversely, if (16) holds and $\{T_n\}$ is asymptotically equivalent to $\{L_n\}$ in the above sense, then (17) also holds.

Proof Covergences (16) and (17) imply that

$$L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow} \left(ne^{h(x)}\right) \sim T_n(x).$$

Take $L_n(x) = g^{-1}(ne^{h(x)})$ and $T_n(x) = f^{-1}(ne^{h(x)})$ where g and f are continuous and strictly increasing functions satisfying (11). Both g^{-1} and f^{-1} are asymptotically inverse to $U(x) = \frac{1}{1-F(x)}$. Since $L_n^{-1}(y) = h^{-1}(\log g(y) - \log n)$ we have

$$\begin{aligned} L_n^{-1} \circ T_n(x) &= h^{-1}\left\{\log \frac{g(T_n(x))}{n}\right\} \\ = h^{-1}\left\{\log \frac{g \circ f^{-1}(ne^{h(x)})}{n}\right\} &\sim h^{-1}\left\{\log \frac{ne^{h(x)}}{n}\right\} = x. \end{aligned}$$

Conversely, (17) can be rewritten as

$$n(1 - F(T_n(x))) = n\left\{1 - F(L_n[L_n^{-1} \circ T_n(x)])\right\} \sim n\left\{1 - F(L_n(x))\right\} \to e^{-h(x)}.$$

$\square$

As a conclusion let us consider the normalization of maxima of normally distributed iid random variables.

Example 5 (Normal distribution.) Let $X_1, X_2, \ldots,$ be iid r.v.s with standard normal d.f.

$$\mathfrak{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \quad \text{and density} \quad \mathfrak{n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in (-\infty, \infty).$$

By Theorem 1 the regular normalizing sequence $L_n(x) = U^{-1}(ne^x)$, where $U = \dfrac{1}{1-\mathfrak{N}}$, causes the weak convergence

$$\mathfrak{N}^n(L_n(x)) \to e^{-e^{-x}}. \tag{18}$$

Theorem 2 says that the rate of this convergence is $O(1/n)$. Unfortunately, the sequence $U^{-1}(ne^x)$ is not very useful in practice, because of the fact that the inverse function $\left(\frac{1}{1-\mathfrak{N}}\right)^{-1}$ is not explicitly known. Thus we go through the well known asymptotic relation

$$U(x) = \frac{1}{1-\mathfrak{N}(x)} \sim g(x) = \frac{x}{\mathfrak{n}(x)} = \sqrt{2\pi} x e^{\frac{x^2}{2}}, \quad x \to \infty \tag{19}$$

in order to find an asymptotic inverse of $g(x)$ and of $U(x)$, respectively. Following the same way as in the proof of (Leadbetter et al. 1983, Theorem 1.5.3) we obtain the following asymptotic inverse of $g(x)$ as $x \to \infty$

$$g^{-1}(x) = \sqrt{2\log x - \log 4\pi - \log\log x}$$

for which

$$U(g^{-1}(x)) \sim g(g^{-1}(x)) \sim x, \quad \text{as } x \to \infty.$$

Now we define the sequence

$$T_n(x) = g^{-1}(ne^x) = \sqrt{2\log n + 2x - \log 4\pi - \log(\log n + x)}.$$

Next we show that both sequences T_n and L_n are asymptotically equivalent in the sense that $L_n^{-1} \circ T_n(x) \to x$ as $n \to \infty$. Indeed, since $L_n^{-1}(y) = \log \frac{U(y)}{n}$ we have

$$L_n^{-1} \circ T_n(x) = \log \frac{U(T_n(x))}{n} = \log \frac{U(g^{-1}(ne^x))}{n} \sim \log \frac{ne^x}{n} = x.$$

Then one can use the sequence $T_n(x)$ for normalization in (18), thus

$$\mathfrak{N}^n(T_n(x)) \to \Lambda(x), \quad n \to \infty. \tag{20}$$

Accordingly to Theorem 2 the rate of convergence in the equation (18) is $O(1/n)$. On the other hand the rate of convergence in the equation (20) depends also on the rate of convergence in the asymptotic relation (19). It is not difficult to show that in this case the rate of convergence is equivalent to that in the liner case, namely $O(\dfrac{1}{\log n})$ (see e.g. (de Haan 1970)).

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