RANDOMLY INDEXED CENTRAL ORDER STATISTICS

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Abstract. In our paper from 2012 we have considered the upper order statistics with central rank of sample with deterministic size. Here we investigate the asymptotic behavior of randomly indexed upper order statistics using regular norming time-space changes.

Keywords: Upper order statistic, Fixed rank, Central rank, Regular norming sequence, Random sample size, Random time change

1. Introduction

On a given probability space $(\Omega, \mathcal{A}, P)$, sufficiently rich, let a point process $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ be defined in the following way:

i) The random arrival process $\{T_k\}$ consists of increasing time points $0 < T_1 < T_2 < ... < T_n \to \infty$. We suppose the inter-arrival times $Y_k := T_k - T_{k-1}$, $k \geq 1$, $T_0 = 0$ are independent rv's, $T_n = \sum\limits_{k=1}^{n} Y_k$. The corresponding counting process $N(t) := \sum\limits_{k} I\{T_k \leq t\} = \max\{k : T_k \leq t\}$ is right continuous.

ii) The random state process $\{X_k\}$ is built by positive iid rv's X_k with continuous df F_X.

iii) Both sequences $\{T_k\}$ and $\{X_k\}$ are independent.

In this initial model, at every fixed moment $t > 0$, we are supplied a sample $\left\{X_1, X_2, ..., X_{N(t)}\right\}$ of random size $N(t)$. Our interest is focused on the upper order statistics (u.o.s.) of this sample

$$X_{N(t):N(t)} < X_{N(t)-1:N(t)} < ... < X_{k:N(t)} < ... < X_{1:N(t)}$$

Definition 1. *We call* $Y_k : \Omega \times (0, \infty) \to (0, \infty)$, $Y_k(t) := X_{k:N(t)}$ *the* k*-th u.o.s. process.*

The asymptotic behavior of $Y_k(t)$ for $t \to \infty$ and k fixed, under linear norming, is considered e.g. in Embrechts, Kluppelberg, Mikosch [1], Chapter

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4.3. For $k=1$ the process $Y_1(t) = \bigvee_{k=1}^{N(t)} X_k$ is an extremal process, investigated e.g. in Balkema, Pancheva [2] and Pancheva [3]. Here "$\bigvee$" denotes the maximum operation.

In this paper we discuss the asymptotic behavior of the k-th u.o.s. process $Y_k(t)$ using monotone normalization. We assume that there exists a sequence of time-space changes on $(0,\infty)\times(0,\infty)$, $\zeta_n(t,x) = (\tau_n(t), u_n(x))$, $\tau_n(t)$ and $u_n(x)$ continuous and strictly increasing, such that the normalized k-th u.o.s. process converges in law to a non-degenerate random process $Y^{(k)}(t)$, i.e.

$$Y_n{}^{(k)}(t) := u_n{}^{-1} \circ Y_k \circ \tau_n(t) = u_n{}^{-1}\left(X_{k:N(\tau_n(t))}\right) \xrightarrow{d} Y^{(k)}(t). \tag{1}$$

Further on we suppose that the norming sequence $\{\zeta_n\}$ is *regular* in the following sense: $\forall s>0$ there exist mappings $\mathcal{U}(s,x)$ and $\mathcal{T}(s,t)$, strictly increasing and continuous in x, respectively in t, such that for $n\to\infty$

$$\lim_{n\to\infty} u_{[ns]}^{-1} \circ u_n(x) = \mathcal{U}(s,x) \tag{2}$$

$$\lim_{n\to\infty} \tau_{[ns]}^{-1} \circ \tau_n(x) = \mathcal{T}(s,x). \tag{3}$$

Moreover, the mappings $s\to\mathcal{U}(s,\cdot) =: \mathcal{U}_s(\cdot)$ and $s\to\mathcal{T}(s,\cdot) =: \mathcal{T}_s(\cdot)$ are one-to-one.

The process $Y_n{}^{(k)}(t)$ is associated with the point process

$$\mathcal{N}_n = \left\{\left(T_{k,n} := \tau_n^{-1}(T_k), X_{k,n} := u_n^{-1}(X_k)\right) : k\ge 1\right\}, n\ge 1.$$

Let $N_n(t) = \max\{k : T_{k,n}\le t\} = \max\{k : T_k \le \tau_n(t)\} = N(\tau_n(t))$ be the counting process of $\mathcal{N}_n$. Consider the u.o.s. of the n-th sample series $\left\{X_{1,n}, X_{2,n}, ..., X_{N(t),n}\right\}$, namely

$$X_{N_n(t):N_n(t),n} < ... < X_{k:N_n(t),n} < ... < X_{1:N_n(t),n}$$

where $X_{k:N_n(t),n} = u_n^{-1}\left(X_{k:N(\tau_n(t))}\right) = Y_n{}^{(k)}(t)$ is the k-th u.o.s. in the n-th sample series of size $N_n(t)$. In this way, with $n\to\infty$, the sample size $N_n(t)$ increases whereas the value of the state points $X_{k,n}$ decreases.

For the limit process $Y^{(k)}(t)$ in (1) we ask the following questions:
Q_1: When does it exist?
Q_2: Which class does it belong to?
The answers depend essentially on the character of the rank k. We consider two cases: k is fixed (Section 2) and $k=k(n)$ is increasing so that $\frac{k(n)}{n}\to$

$\theta \in (0,1)$ (Sections 3). In the latter case one speaks of central order statistics (c.o.s.).

Thus, our main results, Theorems 1 and 3, concern the asymptotic behavior of randomly indexed upper order statistics using regular norming time-space changes.

2. **Fixed rank case**

We start with some preliminaries for extremes. Let $X_1, X_2, ...$ be iid rv's with df F. Assume that there exists a regular (see (2)) norming sequence of space changes, $u_n : (0,\infty) \to (0,\infty)$, continuous and strictly increasing mappings, such that for $n \to \infty$

$$P\left(\bigvee_{k=1}^{n} X_k < u_n(x)\right) = F^n(u_n(x)) \xrightarrow{w} G(x) \tag{4}$$

for $G(x)$ non-degenerate df. Relation (4) is equivalent to $n\overline{F}(u_n(x)) \xrightarrow{w} -\log G(x)$, when $n \to \infty$, for $\overline{F} := 1 - F$. Then the limit df G:

a) satisfies the functional equation

$$G^s(x) = G\left(\mathcal{U}_s(x)\right) \quad \forall s > 0,$$

which determines it as max-stable. The limit mappings in (2), $\{\mathcal{U}_s(\cdot) : s > 0\}$, form a continuous one-parameter group with respect to (w.r.t.) the composition, $\mathcal{U}_s \circ \mathcal{U}_t = \mathcal{U}_{st}$;

b) has the explicit form $G(x) = \exp\left\{-e^{-g(x)}\right\}$ with $g : \text{supp}G \to (-\infty,\infty)$ continuous and strictly increasing and $\mathcal{U}_s(x) = g^{-1}\left(g(x) - \log s\right)$ (see Pancheva [4]). Here supp$G$ is an abbreviation for the support of G.

If (4) holds we say that F belongs to the general max-domain of attraction of G, briefly $F \in \max -\mathcal{DA}(G)$.

Let us return to the initial model of Section 1. Denote by $X_{k:n}$ the k-th u.o.s. of the sample $\{X_1, X_2, ..., X_n\}$ with a continuous df F_X. The asymptotic behavior of the normalized $X_{k:n}$ is stated in the following

Proposition 1. *Suppose $F_X \in \max -\mathcal{DA}$ of a max-stable df G w.r.t. a regular norming sequence $\{u_n(\cdot)\}$. Then for fixed k and $n \to \infty$*

$$P\left(X_{k:n} < u_n(x)\right) \xrightarrow{w} H(x) = \overline{\Gamma}_k\left(-\log G(x)\right),$$

where $\Gamma_k(x) = \frac{1}{(k-1)!}\int\limits_0^x t^{k-1}e^{-t}dt$ is the Gamma df.

The proof of Proposition 1 is similar to the proof of Theorem 4.2.3 and Corollary 4.2.4 in Embrechts, Kluppelberg, Mikosch [1].

When using linear norming mappings $u_n(x) = a_n x + b_n$, $a_n > 0$, b_n real, the limit df G in (4) belongs to one of the three types of the well known extreme value distributions. In the more general setting of monotone normalization, using regular norming sequences $\{u_n(\cdot)\}$, the limit df in (4) might be any continuous and strictly increasing df G.

Before analyzing the limit class of the processes obtained in (1), we need to agree on the asymptotic behavior of the counting process $N_n(t)$. Let us assume that $\forall t > 0$

$$\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t. \tag{5}$$

Here Λ_t with df F_Λ is random time change.

Definition 2. *Under random time change we understand a strictly increasing and continuous random function* $\Lambda : (0,\infty) \to (0,\infty)$, $\Lambda(0) = 0$ *and* $\Lambda(t) \to \infty$ *for* $t \to \infty$.

Now we are ready to formulate our first result on randomly indexed u.o.s.

Theorem 1. *Suppose* $F_X \in \max -\mathcal{DA}(G)$ *w.r.t. a regular norming sequence* $\{u_n\}$. *Assume that the counting process* $N_n(t)$ *satisfies* (5). *Then for fixed* k *and* $n \to \infty$

$$P\left(X_{k:N_n(t)} < u_n(x)\right) \xrightarrow{w} \int_0^\infty H\left(\mathcal{U}(s,x)\right) dF_{\Lambda(t)}(s) = EH \circ \mathcal{U}(\Lambda_t, x)$$

where $H(x) = \overline{\Gamma}_k(-\log G(x))$ *is the limit distribution from Proposition 1 and* $\mathcal{U}(s,x) = \lim_{n\to\infty} u^{-1}_{[ns]} \circ u_n(x)$.

To keep the paper short we have to omit the proof, but let us mention that it is similar to the proof of Theorem 3 below and uses Dini's theorem (see Rudin [5]), Dominated convergence theorem and the following analytical result.

Lemma 1. *Assume* $\{Q_n\}$ *is a sequence of dfs on* $(0,\infty)$ *such that* $Q_n \longrightarrow Q$ *uniformly. Let* $\{f_n\}$ *be a sequence of functions on* $(0,\infty)$ *converging to* f Q*-almost everywhere. Suppose* $|f_n(z)| \le 1$. *Then* f *is* Q*-integrable and*

$$\lim_{n\to\infty} \int_0^\infty f_n(z) dQ_n(z) = \int_0^\infty f(z) dQ(z).$$

Corollary to Theorem 1. Let $k = 1$, then

$$P\left(\bigvee_{j=1}^{N_n(t)} X_j < u_n(x)\right) \xrightarrow{w} EG \circ \mathcal{U}(\Lambda_t, x) = EG^{\Lambda_t}(x).$$

Theorem 1 gives answer to our first question Q_1: If $F^n\left(u_n(x)\right) \xrightarrow{w} G(x)$, where $\{u_n\}$ is regular, and if $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$, Λ_t random time change, then there exists a random process $Y^{(k)}(t)$ with df $P\left(Y^{(k)}(t) < x\right) =: g(t,x) = E\overline{\Gamma}_k\left(-\log G\left(\mathcal{U}(\Lambda_t, x)\right)\right)$, such that $Y_n^{(k)}(t) = u_n^{-1}\left(X_{k:N_n(t)}\right) \xrightarrow{d} Y^{(k)}(t)$.

Remark. In the asymptotic results for randomly indexed samples with size N_n, known in the literature, the authors usually suppose convergence in probability $\frac{N_n}{n} \xrightarrow{P} \Lambda$, Λ positive rv, e.g. Galambos [6], Theorem 6.2.1, Bilingsley [7], Theorem 17.2. In our model, we assume the sequences $\{N_n\}$ and $\{X_n\}$ are independent and Λ is random time change. Thus, it is enough to suppose a convergence in distribution $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$.

Next, using the regularity of the norming sequence $\{\tau_n\}$ (see (3)), we give an answer to question Q_2. Let $\stackrel{fdd}{=}$ denote equivalence of the finite dimensional distributions.

Theorem 2. *The limit process $Y^{(k)}$ in (1) is self-similar w.r.t. the continuous one-parameter group $\{\eta_s(t,x) = (\mathcal{T}_s(t),\mathcal{U}_s(x)) : s > 0\}$ of time-space changes, i.e.*

$$Y^{(k)}(t) \stackrel{fdd}{=} \mathcal{U}_s^{-1} \circ Y^{(k)} \circ \mathcal{T}_s(t),\ \forall s > 0.$$

Proof. Observe that $Y^{(k)}(t) \stackrel{d}{=} \mathcal{U}_s^{-1} \circ Y^{(k)} \circ \mathcal{T}_s(t)$ is equivalent to $g(t,x) = g\left(\mathcal{T}_s(t),\mathcal{U}_s(x)\right)$. For $s > 0$, by Theorem 1,

$P\left(X_{k:N_n(t)} < u_{[ns]}(x)\right) = P\left(X_{k:N_n(t)} < u_n\left(u_n^{-1} \circ u_{[ns]}(x)\right)\right)$

$\xrightarrow{w} EH \circ \mathcal{U}\left(\Lambda_t,\mathcal{U}^{-1}(s,x)\right) = g\left(t,\mathcal{U}_s^{-1}(x)\right)$.

On the other hand, substituting $m = [ns]$ and $\gamma = 1/s$, we get

$P\left(X_{k:N_n(t)} < u_{[ns]}(x)\right) = P\left(X_{k:N\left(\tau_{[m\gamma]}(t)\right)} < u_m(x)\right)$

$\xrightarrow{w} EH \circ \mathcal{U}\left(\Lambda\left(\mathcal{T}_s(t), x\right)\right) = g\left(\mathcal{T}_s(t), x\right)$.

In the last step we have used that $N\left(\tau_{[m\gamma]}(t)\right) = N\left(\tau_m\left(\tau_m^{-1} \circ \tau_{[m\gamma]}(t)\right)\right)$ and $\tau_m^{-1} \circ \tau_{[m\gamma]}(t) \longrightarrow \mathcal{T}^{-1}(\gamma,t) = \mathcal{T}(s,t)$, for $m \to \infty$. Comparing both limit relations for $P\left(X_{k:N_n(t)} < u_{[ns]}(x)\right)$, we conclude the statement. □

3. Increasing rank case

In this section we consider $Y_n^{(k_n)}(t) = u_n^{-1} \circ X_{k_n:N_n(t)}$, the central u.o.s. process, where the rank $k = k_n$ increases with n in such a way that

$$\frac{k_n}{n} \to \theta \in (0,1)\,. \tag{6}$$

Further we assume that (5) holds, i.e. $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$, Λ_t random time change.

We ask for the asymptotic behavior of $Y_n^{(k_n)}(t)$ as $n \to \infty$. Let us preliminary consider a sample with non-random size l_n, $l_n \to \infty$ as $n \to \infty$, namely $\{X_1, X_2, ..., X_{l_n}\}$. We form the c.o.s. $X_{k_n:l_n}$ with the property $\frac{k_n}{l_n} \to \lambda \in (0,1)$. As norming mappings we again use the regular sequence $\{u_n(\cdot)\}$ of space changes $u_n : (0,\infty) \to (0,\infty)$. Denote $p_n(\cdot) = P(X_i \geq u_n(\cdot))$. Just analogously to Theorem 2.5.2 in Leadbetter, Lindgren, Rootzen [8], where $u_n(x)$ are linear, one can prove the following statement:

Proposition 2. *Suppose $k_n \to \infty$, $k_n < l_n$, $\frac{k_n}{l_n} \to \lambda \in (0,1)$ and $l_n(1 - p_n(\cdot))p_n(\cdot) \to \infty$, as $n \to \infty$. Let*

$$\frac{k_n - l_n p_n(x)}{\sqrt{l_n p_n(x)(1 - p_n(x))}} \xrightarrow{w} \tau_\lambda(x). \tag{7}$$

Then for $n \to \infty$

$$P\left(X_{k_n:l_n} < u_n(x)\right) \xrightarrow{w} \Phi\left(\tau_\lambda(x)\right) \tag{8}$$

where $\Phi(x)$ is the standard normal df.

In order to obtain a unique limit distribution in (8), usually one assumes the second order condition $\sqrt{n}(\frac{k_n}{l_n} - \lambda) \to 0$, for $n \to \infty$.

An equivalent version of Proposition 2 is proved in Pancheva, Gacovska [9]. Moreover, there we have shown that $\tau(\theta, x) := \tau_\theta(x)$ satisfies the functional equation

$$\sqrt{z} \cdot \tau(\theta, x) = \tau\left(\theta, \mathcal{U}_z(x)\right) \tag{9}$$

$\forall z > 0$, x continuity points of τ and $\mathcal{U}_z(x)$ from (2). As a consequence one deduces that the limit class in (8) contains thirteen possible types.

More results on the asymptotic behavior of randomly indexed ordered statistics can be found e.g. in Barakat, El-Shandidy [10] and in Surkov [11].

Let us return to the randomly indexed c.o.s. $X_{k_n:N_n(t)}$ satisfying (5) and (6). On the basis of the previous Proposition 2, we now state the main result:

Theorem 3. *Suppose $k_n \to \infty$ and (6) is satisfied. Assume there exists a regular norming sequence $\{u_n(\cdot)\}$ such that $n\overline{F}(u_n(x))F(u_n(x)) \to \infty$ and*

$$\sqrt{n}\frac{\theta - \overline{F}(u_n(x))}{\sqrt{\theta(1-\theta)}} \xrightarrow{w} \tau_\theta(x), \tag{10}$$

for $n \to \infty$. If additionally (5) holds true, then $\forall t > 0$

$$u_n^{-1} \circ X_{k_n:N_n(t)} \xrightarrow{d} Y_0(t)$$

where the limit process has df

$$g_0(t,x) = \int_\theta^\infty \Phi \circ \tau\left(\tfrac{\theta}{z}, \mathcal{U}_z(x)\right) dF_{\Lambda(t)}(z) = E[\Phi \circ \tau\left(\theta {\Lambda_t}^{-1}, \mathcal{U}_{\Lambda(t)}(x)\right) I\{\Lambda_t > \theta\}]$$

and $\mathcal{U}(\cdot,\cdot)$ is defined in (2).

Proof. Formally written

$P(X_{k_n:N_n(t)} < u_n(x)) = \int\limits_0^\infty P\left(X_{k_n:N_n(t)} < u_n(x) \mid N_n(t) = s\right) dP\left(\frac{N_n(t)}{n} = \frac{s}{n}\right)$

and substituting here $\frac{s}{n} = z$ we continue with

$= \int\limits_{A_n} P\left(X_{k_n:[nz]} < u_n(x)\right) dP\left(\frac{N_n(t)}{n} = z\right),$

where $A_n = \{z : z > \frac{k_n}{n}\}$. Obviously $A_n \to A = \{z : z \geq \theta\}$ and consequently $A_n \bigcap A \to A$. In order to apply Proposition 2, we have to check if condition (7) holds for $l_n = [nz]$, $p_n(x) = \overline{F}(u_n(x))$. Denote by A^o the interior of A. Then, for $z \in A_n \bigcap A^o$, just as in the proof of Theorem 2.5.1 in Leadbetter, Lindgren, Rootzen [8], we observe that for $n \to \infty$

$$\frac{k_n - [nz]\overline{F}(u_n(x))}{\sqrt{[nz]\overline{F}(u_n(x))F(u_n(x))}} \sim \sqrt{z} \cdot \sqrt{n}\frac{\frac{\theta}{z} - \overline{F}(u_n(x))}{\sqrt{\frac{\theta}{z}\cdot\left(1-\frac{\theta}{z}\right)}} \xrightarrow{w} \sqrt{z}\cdot\tau\left(\frac{\theta}{z}, x\right).$$

The last consequence is due to condition (10). Since τ satisfies the functional equation (9), for all x continuity points of τ and $z > \theta$, we have

$$\sqrt{z}\cdot\tau\left(\tfrac{\theta}{z}, x\right) = \tau\left(\tfrac{\theta}{z}, \mathcal{U}_z(x)\right).$$

We attend to apply Lemma 1 to $\int\limits_{A_n} P\left(X_{k_n:[nz]} < u_n(x)\right) dP\left(\frac{N_n(t)}{n} = z\right)$. The last integral is of the form $\int\limits_{A_n} f_n(z) dQ_n(z)$ where for $n \to \infty$,

$$Q_n(z) := P\left(\frac{N_n(t)}{n} < z\right) = \sum_{j=1}^{nz} P\left(N_n(t) = j\right) \xrightarrow{w} P\left(\Lambda_t < z\right) =: Q(z),$$
$$f_n(z) := P\left(X_{k_n:[nz]} < u_n(x)\right) = \sum_{j=0}^{k_n-1} \binom{[nz]}{j} \overline{F}^j(u_n(x))F^{[nz]-j}(u_n(x))$$
$$= P\left(X_{k_n:[nz]} < u_{[nz]}\left(u^{-1}_{[nz]} \circ u_n(x)\right)\right) \xrightarrow{w} \Phi \circ \tau\left(\tfrac{\theta}{z}, \mathcal{U}_z(x)\right) =: f(z).$$

The last convergence holds for all $z > \theta$ and x continuity points of τ, by Proposition 2. Note, in fact f is defined on A^o and f_n is defined on the set A_n. Further more, for any $z \in A^o$ there exists a number $n_0 = n_0(z)$, such that $z \in A_n \bigcap A^o$ for all $n > n_0$. Lemma 1 guarantees the convergence of $\int f_n(z)dQ_n(z) \longrightarrow \int f(z)dQ(z)$ on a set where the convergence $Q_n \longrightarrow Q$ is uniform. We observe that $Q_n(z)$ is continuous in z and converges monotonically in n to the continuous $Q(z)$. By Dini's theorem (see Rudin [5]), the convergence $Q_n \longrightarrow Q$ is uniform on compact subsets of A^o.

Recall, $\tau(x)$ has at the most three jumps, at the left end point of the supp$H$, at the right end point of the supp$H$ and at the median of H, where $H = \Phi \circ \tau$ (c.f. Pancheva, Gacovska [9]). Since Q is continuous, we have $f_n \longrightarrow f$ Q-almost everywhere on the set A^o.

Since A^o can be approximated by compact subsets of A, we can apply Lemma 1 in a slightly different form

$$\lim_{n\to\infty} \int_{A^o} f_n dQ_n = \int_{A^o} f dQ.$$

Returning to the previous notation, we gain the final statement

$$\lim_{n\to\infty} P(X_{k_n:N_n(t)} < u_n(x)) = \lim_{n\to\infty} \int_{A^o} P\left(X_{k_n:[nz]} < u_n(x)\right) dP\left(\tfrac{N_n(t)}{n} = z\right)$$
$$= \int_{\theta}^{\infty} \Phi \circ \tau\left(\tfrac{\theta}{z}, \mathcal{U}_z(x)\right) dF_{\Lambda(t)}(z)$$
$$= E[\Phi \circ \tau\left(\theta{\Lambda_t}^{-1}, \mathcal{U}_{\Lambda(t)}(x)\right) I\left\{\Lambda_t > \theta\right\}] = g_0(t, x). \qquad \square$$

Theorem 3 gives an answer to question Q_1 for the case of increasing rank k_n. In order to give an answer to question Q_2 we still have to characterize the limit process Y_0. Analogously to Theorem 2 one can show the following

Theorem 4. *The limit process Y_0 is self-similar w.r.t. the continuous one-parameter group $\{\eta_s(t,x) = (\mathcal{T}_s(t), \mathcal{U}_s(x)) : s > 0\}$, so for $t > 0$*

$$g_0(t, x) = g_0\left(\mathcal{T}_s(t), \mathcal{U}_s(x)\right), \quad \forall s > 0. \tag{11}$$

An important and useful consequence of the selfsimilarity of a random process Y, i.e. $\mathcal{U}_s \circ Y(t) \stackrel{d}{=} Y \circ \mathcal{T}_s(t)$, is the possibility to determine its fdd's knowing the df $g(1,x) := P(Y(1) < x)$ only. Indeed, denote by $s(t)$ the unique solution of $\mathcal{T}_s(t) = 1$. Consequently $Y(t) \stackrel{d}{=} \mathcal{U}_{s(t)}^{-1} \circ Y(1)$, $\forall t > 0$. Then

$$P(Y(t_1) < x_1, ..., Y(t_k) < x_k) = P(Y(1) < min(\mathcal{U}_{s(t_1)}(x_1), ..., \mathcal{U}_{s(t_k)}(x_k))).$$

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