Improved bounds for bounded occurrence constraint satisfaction

Johan Håstad
Royal Institute of Technology

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Abstract

We show how to improve the dependence on the number of occurrences of a variable when approximating CSPs. The result applies when the interesting part of the predicate is odd and says that the advantage is $\Omega(D^{-1/2})$ if each variable appears in at most D terms.

1 History of this paper

The result of this paper was obtained independently but after the paper by Barak et al [1]. The underlying ideas of the two proofs are essentially the same but as this note was written and the presentations of [1] and this note are quite different I decided to make a very primitive version of this note available to the public. For a proper academic paper discussion of the problem and its history we refer to [1].

2 The argument

Let the multilinear expansion of the objective function be

$$f(x) = \sum f_\alpha x^\alpha$$

where each term is of degree at most k and each variable appears in at most D terms. We have the following theorem.

Theorem 2.1 *There is a constant d_k only depending on k such that in probabilistic polynomial time it is possible to find an assignment x such*

$$|f(x)| \geq d_k D^{-1/2} \sum |f_\alpha|.$$

We suspect that the methods discussed in [1] are sufficient to derandomize also the algorithm in the current paper.

Proof: We first give the algorithm.

1. Select a random set, S, of the variables by including each variable in S with probability one half. For notational convenience we rename the variable x_i to y_i if it is placed in S.

2. Select uniformly random values, y^0 for the variables of S. Let $g(x) = f(x, y^0)$ be the induced function in the remaining variables.

3. Write $g(x) = g^0(y^0) + \sum_i x_i g^i(y^0) + G^2(x, y^0)$ where G^2 contains all terms that remain at least degree two in the x-variables.

4. Find a suitable parameter t and set x_i to the sign of $g^i(y^0)$ with probability $(1+t)/2$ for all i independently.

5. If the value obtained is not large enough repeat from step 1.

We return how to find a suitable parameter t later. Let us analyze this procedure. We say that a set α is *good* if it contains exactly one element of S. In expectation we have

$$E\left[\sum_{\alpha \text{ } good} |f_\alpha|\right] = k2^{-k} \sum_\alpha |f_\alpha|.$$

The good sets naturally fall in the classes N_i where $\alpha \in N_i$ if it contains only x_i of the variables not in S. It is not difficult to see that conditioned on the first two steps the expected value of $f(x, y^0)$ is a degree k polynomial, $Q(t)$. If we let q_1 denote the coefficient of the linear term we have $q_1 = \sum |g^i(y^0)|$. We claim that

$$E\left[|g_i(y^0)|\right] \geq c_k E\left[(g_i(y^0))^2\right]^{1/2} = c_k \left(\sum_{\alpha \in N_i} f_\alpha^2\right)^{1/2} \geq D^{-1/2} c_k \sum_{\alpha \in N_i} |f_\alpha|, \quad (1)$$

for a constant c_k depending only on k. Indeed the first inequality is just saying $\|g_i\|_1 \geq c_k \|g_i\|_2$ which is true as all L^p-norms are comparable for degree $k-1$ polynomials. The last step follows from Cauchy-Schwarz inequality as

$$\sum_{\alpha \in N_i} |f_\alpha| \leq \left(\sum_{\alpha \in N_i} 1\right)^{1/2} \left(\sum_{\alpha \in N_i} f_\alpha^2\right)^{1/2} \leq \sqrt{D} \left(\sum_{\alpha \in N_i} f_\alpha^2\right)^{1/2}.$$

As the union of all N_i give all the good sets we conclude from (1) that

$$E[q_1] = E\left[\sum_i |g_i(y)|\right] \geq c_k D^{-1/2} E\left[\sum_{\alpha \text{ } good} |f_\alpha|\right] \geq k2^{-k} c_k D^{-1/2} \sum_\alpha |f_\alpha|. \quad (2)$$

It is easy to see that $q_1 \leq \sum_\alpha |f_\alpha|$ and thus with probability at least $\frac{1}{2}k2^{-k}c_k D^{-1/2}$ we have

$$q_1 \geq \frac{1}{2}k2^{-k}c_k D^{-1/2} \sum_\alpha |f_\alpha|.$$

Now recall Markov brothers' inequality which says that if P is a polynomial of degree k then

$$\max_{t\in[-1,1]} |P(t)| \geq c'_k P'(0),$$

where c'_k is an explicit and known constant. This implies that there is a value of t such that that

$$|Q(t)| \geq \frac{1}{2}k2^{-k}c'_k c_k D^{-1/2} \sum_\alpha |f_\alpha|.$$

Once y^0 is chosen, the polynomial Q is completely explicit and hence it is possible to find (a very good approximation of) the optimal t. In particular it is possible, in polynomial time, to find a value of t which satisfies $|Q(t)| \geq \frac{1}{3}k2^{-k}c'_k c_k D^{-1/2} \sum_\alpha |f_\alpha|$. Setting $d_k = \frac{1}{4}k2^{-k}c'_k c_k$ it is easy to see that the algorithm succeeds with a good probability. ∎

Let us end with some brief comments. First observe that there are several ways to choose a suitable t. As one alternative one could use a uniformly random t and use an inequality of the form then

$$E_{t\in[-1,1]}[|P(t)|] \geq c''_k P'(0),$$

which is clearly true for some value of c''_k even if I do not know a reference and the best value for c''_k. As a further alternative [1] uses extrema of Chebychev polynomials as a set of possible choices for t.

We might be interested in finding an assignment such that $f(x)$ is large and positive. If f is odd this is easy since we can simply negate an x giving a large negative value. In the general case this is not possible and indeed, as also observed in [1], this is a real problem as can be seen from the following example.

Let us take Max-Cut where we score 2 for cut edges and -2 for not cut edges. Consider the complete graph on $D+1$ vertices. We have the objective function

$$-2\sum_{i>j} x_i x_j = (\sum x_i^2) - (\sum_i x_i)^2 \leq D+1$$

and thus the global optimum is at most $D+1$. On the other hand the sum of the absolute values of all coefficients is $\Omega(D^2)$ (and the global maximum of the absolute value is also $\Omega(D^2)$). This implies that to get a general result for arbitrary predicates that we can beat the trivial approximation ratio by an additional term $\Omega(D^{-1/2})$ we must use that fact the global maximum is large. Thus something new is needed.

References

[1] B. Barak, A. Moitra, R. O'Donnell, P. Raghavendra, O. Regev, D. Steurer, L. Trevisan, A. Vijayaraghaven, D. Witmer, and J. Wright, Beating the random assignment on constraint satisfaction problems of bounded degree. *Arxiv 1505.03424*.