Towards an Understanding of Polynomial Calculus: New Separations and Lower Bounds

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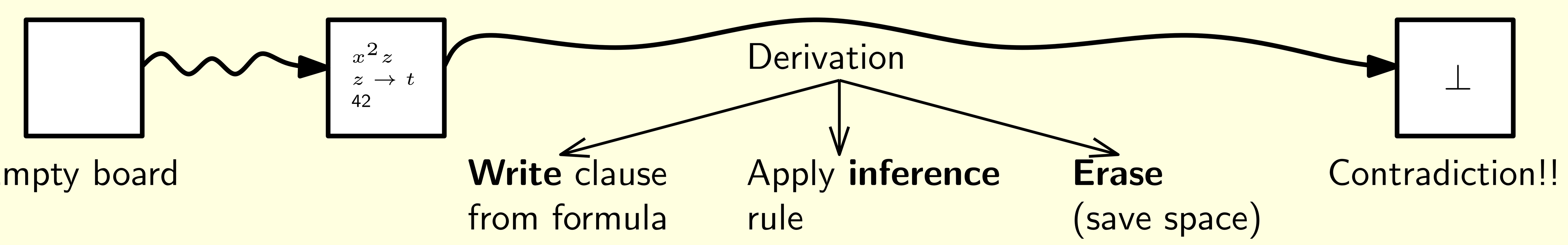
Refuting CNF formulas

Proof: sequence of whiteboards



Interesting measures:
Size: $\approx$ # boards
Space: $\approx$ Largest board

Resolution

Lines are clauses, e.g. $x \vee y \vee \bar{z}$, inference rules are:

$$\frac{C \vee x \qquad D \vee \bar{x}}{C \vee D}$$

Size = # clauses, space = # clauses on largest board
Auxiliary measure: width = size of largest clause

Polynomial Calculus (PC)

Lines are polynomials, e.g. $x\bar{y} + 2z$, roots denote truth, inference rules are: $\frac{p \qquad q}{\alpha p + \beta q}$, $\frac{p}{xp}$

Size = # monomials, space = # monomials on largest board
Auxiliary measure: degree of largest monomial

Previous results

Resolution			PC
Large width	$\Longleftrightarrow$	Large size	Large degree $\Longleftrightarrow$ Large size
Large width	$\Longrightarrow$	Large space	???
Small width	$\not\Longrightarrow$	Small space	???

Ingredients

XOR substitution

$$C = x \vee \bar{y}$$

$$x \mapsto x_1 \oplus x_2$$

$$\begin{aligned} C[\oplus] &= x_1 \vee x_2 \vee \bar{y}_1 \vee y_2 \\ &\wedge \bar{x}_1 \vee \bar{x}_2 \vee \bar{y}_1 \vee y_2 \\ &\wedge x_1 \vee x_2 \vee y_1 \vee \bar{y}_2 \\ &\wedge \bar{x}_1 \vee \bar{x}_2 \vee y_1 \vee \bar{y}_2 \end{aligned}$$

Space framework

PC space lower bounds from combinatorial game

Introduced by [Bonacina & Galesi '13]

Implies all space results known to date
But fails to prove some believable results

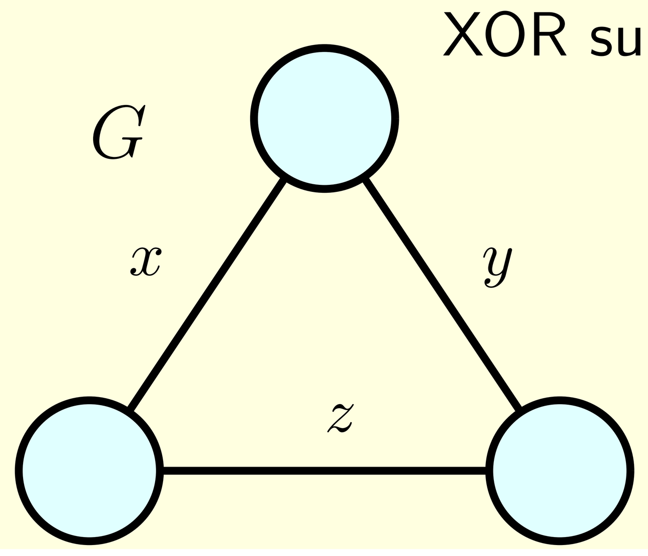
Open: characterize space?

Tseitin Formulas

Variables are edges
Odd parity at each vertex

Falsify the even handshakes principle:
Sum of edge parities even

XOR substitution same as double edges



$$\begin{aligned} \mathrm{Ts}(G) &= x \vee y \;\wedge\; \bar{x} \vee \bar{y} \\ &\wedge\; y \vee z \;\wedge\; \bar{y} \vee \bar{z} \\ &\wedge\; x \vee z \;\wedge\; \bar{x} \vee \bar{z} \end{aligned}$$

Our results: relating PC space and degree

Large width $\Longrightarrow$ Large space of $F[\oplus]$

PC space of refuting $F[\oplus] \gtrsim$ Resolution width of refuting F
Not quite Large degree $\Longrightarrow$ Large space
Stronger because Large degree $\Longrightarrow$ Large width
Weaker because XOR substitution changes F a lot

Open: Large degree $\overset{?}{\Longrightarrow}$ Large space of F

Small degree $\not\Longrightarrow$ Small space

G expander graph with double copies of each edge
Then Tseitin formula $\mathrm{Ts}(G)$ has:
PC **degree**: constant (minimum)
PC **space**: linear (maximum)

Open: tight bound for non-multigraphs?

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