The Size of Coefficients in Cutting Planes Proofs

Marc Vinyals

KTH Royal Institute of Technology
Stockholm, Sweden

joint work with Susanna F. de Rezende and Jakob Nordström

Cutting Planes

Work with inequalities

$$x \vee \overline{y} \quad \rightarrow \quad x + (1 - y) \geq 1 \quad \rightarrow \quad x - y \geq 0$$

Cutting Planes

Work with inequalities

$$x \lor \overline{y} \quad \to \quad x + (1 - y) \geq 1 \quad \to \quad x - y \geq 0$$

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (a_i + b_i) x_i \geq a + b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

Cutting Planes

Work with inequalities

$$x \lor \overline{y} \quad \to \quad x + (1 - y) \geq 1 \quad \to \quad x - y \geq 0$$

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (a_i + b_i) x_i \geq a + b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

Goal: derive $0 \geq 1$

CP in Practice

Question

How can we implement CP *in practice?*

CP in Practice

Question

How can we implement CP *in practice?*

- When do we add inequalities?
- How important is division?
- Do we need exponential coefficients?

CP in Practice

Question

How can we implement CP *in practice?*

- When do we add inequalities?
- How important is division?
- Do we need exponential coefficients?

CP Simulates Resolution

$$\frac{x \vee y \vee z \qquad \overline{x} \vee y \vee \overline{t}}{y \vee z \vee \overline{t}}$$

$$\cfrac{\cfrac{x + y + z \geq 1 \qquad -x + y - t \geq -1}{2y + z - t \geq 0}}{\cfrac{2y + 2z - 2t \geq -1}{y + z - t \geq 0}}$$

CP Simulates Resolution

$$\frac{x \vee y \vee z \qquad \bar{x} \vee y \vee \bar{t}}{y \vee z \vee \bar{t}}$$

$$\cfrac{\cfrac{\cfrac{x + y + z \geq 1 \qquad -x + y - t \geq -1}{2y + z - t \geq 0}}{2y + 2z - 2t \geq -1}}{y + z - t \geq 0}$$

- Length increases at most a factor n
- Coefficients 2 enough

Separation of CP and Resolution

Pigeonhole principle:

- $\bigvee_{j=1}^{n} x_{ij}$ for each pigeon
- $\{\overline{x_{ij}} \vee \overline{x_{ij'}}\}_{j \neq j'}$ for each hole

Separation of CP and Resolution

Pigeonhole principle:

- $\bigvee_{j=1}^{n} x_{ij}$ for each pigeon
- $\{\overline{x_{ij}} \vee \overline{x_{ij'}}\}_{j \neq j'}$ for each hole

1. Derive $\sum_{i=1}^{n+1} x_{ij} \leq 1$ for each hole

$$\dfrac{\dfrac{-x_{11} - x_{21} \geq -1 \qquad -x_{11} - x_{31} \geq -1 \qquad -x_{21} - x_{31} \geq -1}{-2x_{11} - 2x_{21} - 2x_{31} \geq -3}}{-x_{11} - x_{21} - x_{31} \geq -1}$$

2. Add all inequalities

$$\dfrac{\{\sum_{j=1}^{n} x_{ij} \geq 1\}_{i=1}^{n+1} \qquad \{\sum_{i=1}^{n+1} -x_{ij} \geq -1\}_{j=1}^{n}}{0 \geq 1}$$

Separation of CP and Resolution

Pigeonhole principle:

- $\bigvee_{j=1}^{n} x_{ij}$ for each pigeon
- $\{\overline{x_{ij}} \vee \overline{x_{ij'}}\}_{j \neq j'}$ for each hole

1. Derive $\sum_{i=1}^{n+1} x_{ij} \leq 1$ for each hole

$$\dfrac{\dfrac{-x_{11} - x_{21} \geq -1 \qquad -x_{11} - x_{31} \geq -1 \qquad -x_{21} - x_{31} \geq -1}{-2x_{11} - 2x_{21} - 2x_{31} \geq -3}}{-x_{11} - x_{21} - x_{31} \geq -1}$$

2. Add all inequalities

$$\dfrac{\{\sum_{j=1}^{n} x_{ij} \geq 1\}_{i=1}^{n+1} \qquad \{\sum_{i=1}^{n+1} -x_{ij} \geq -1\}_{j=1}^{n}}{0 \geq 1}$$

- Coefficients 2 enough

Exponential Lower Bounds

Clique vs coloring formula: “There is

- a set of edges E,
- a mapping $c : [k] \to V$ such that $c([k])$ is a k-clique, and
- a $k-1$-coloring $\chi : V \to [k-1]$.”

Exponential Lower Bounds

Clique vs coloring formula: “There is

- a set of edges E,
- a mapping $c : [k] \to V$ such that $c([k])$ is a k-clique, and
- a $k-1$-coloring $\chi : V \to [k-1]$.”

Needs exponential length in CP. Proof:

1. Interpolation $\to$ monotone circuit for k-clique
2. Lower bound for monotone circuits

Exponential Lower Bounds

Clique vs coloring formula: "There is

- a set of edges E,
- a mapping $c : [k] \to V$ such that $c([k])$ is a k-clique, and
- a $k-1$-coloring $\chi : V \to [k-1]$."

Needs exponential length in CP. Proof:

1. Interpolation $\to$ monotone circuit for k-clique
2. Lower bound for monotone circuits

- First proved for CP* [Bonet, Pitassi, Raz '95]
- Proof for CP uses lower bound for real circuits [Pudlák '97]

Weak Division

- Practical solvers use weaker division rule.
- Can solve PHP in polynomial time if properly encoded.

Weak Division

- Practical solvers use weaker division rule.
- Can solve PHP in polynomial time if properly encoded.

Theorem

Resolution simulates CP^* *with weak division starting from CNF.*

What about unbounded coefficients?

Small Space

Line space: max # inequalities in memory

Small Space

Line space: max # inequalities in memory

Theorem ([Galesi, Pudlák, Thapen '15])

Every formula has a CP *proof in line space* 5.

- Exponential coefficients
- Exponential length

Small Space

Line space: max # inequalities in memory

Theorem ([Galesi, Pudlák, Thapen '15])

Every formula has a CP proof in line space 5.

- Exponential coefficients
- Exponential length

Theorem ([Galesi, Pudlák, Thapen '15])

There is a formula that requires line space $\Omega(\log\log\log n)$ in $\mathsf{CP^k}$.

Separation of CP and Resolution

Can we separate resolution and CP* space?

Theorem ([Galesi, Pudlák, Thapen '15])

PHP has a CP^2 *proof in line space* 5.

Separation of CP and Resolution

Can we separate resolution and CP^* space?

Theorem ([Galesi, Pudlák, Thapen '15])

PHP has a CP^2 *proof in line space* 5*.*

Can we separate resolution and CP^* space for easy formulas?

Theorem

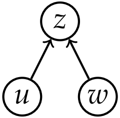
There is a family of 9*-CNFs of* n *variables and size* $\mathrm{O}(n)$ *such that*

- *There are* CP^2 *proofs of length* $\mathrm{O}(n)$ *and line space* $\mathrm{O}(1)$
- *There are resolution proofs of length* $\mathrm{O}(n)$
- *Resolution proofs require line space* $\Omega(\sqrt{n})$

Pebbling Formulas

Graph G given. Formula Peb_G defined as:

- For each vertex v, a variable v.
- For each source s, a constraint
  s.
- For each non-source v with preds u and w, a constraint
  $u \wedge w \rightarrow v$
- For the sink z, the constraint
  $\bar{z}$.



u

w

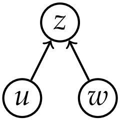
$\overline{u} \vee \overline{w} \vee z$

$\overline{z}$

Substituted Pebbling Formulas

Graph G given. Formula $\mathrm{Peb}_G[\geq]$ defined as:

- For each vertex v, 3 variables v_1, v_2, v_3.
- For each source s, a constraint $s_1 + s_2 + s_3 \geq 2$.
- For each non-source v with preds u and w, a constraint
  $[u_1 + u_2 + u_3 \geq 2] \wedge$
  $[w_1 + w_2 + w_3 \geq 2] \rightarrow$
  $[v_1 + v_2 + v_3 \geq 2]$
- For the sink z, the constraint $z_1 + z_2 + z_3 \leq 1$.

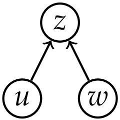


$$u_1 \vee u_2 \quad u_1 \vee u_3 \quad u_2 \vee u_3$$
$$w_1 \vee w_2 \quad w_1 \vee w_3 \quad w_2 \vee w_3$$
$$\overline{u_1} \vee \overline{u_2} \vee \overline{w_1} \vee \overline{w_2} \vee z_1 \vee z_2$$
$$\overline{u_1} \vee \overline{u_3} \vee \overline{w_1} \vee \overline{w_2} \vee z_1 \vee z_2$$
$$\vdots$$
$$\overline{u_2} \vee \overline{u_3} \vee \overline{w_2} \vee \overline{w_3} \vee z_2 \vee z_3$$
$$\overline{z_1} \vee \overline{z_2} \quad \overline{z_1} \vee \overline{z_3} \quad \overline{w_2} \vee \overline{w_3}$$

Threshold Pebbling Formulas

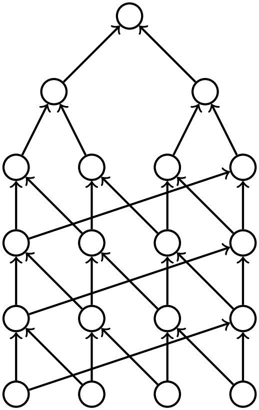
Graph G given. Formula $\mathrm{Peb}_G[T]$ defined as:

- For each vertex v, 3 variables v_1, v_2, v_3.
- For each source s, a constraint $s_1 + s_2 + s_3 \geq 2$.
- For each non-source v with preds u and w, a constraint $u_1 + u_2 + u_3 + w_1 + w_2 + w_3 \leq 2(v_1 + v_2 + v_3)$
- For the sink z, the constraint $z_1 + z_2 + z_3 \leq 1$.

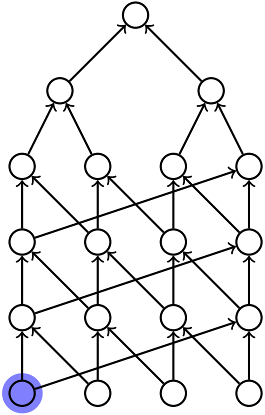


$$u_1 \vee u_2 \quad u_1 \vee u_3 \quad u_2 \vee u_3$$
$$w_1 \vee w_2 \quad w_1 \vee w_3 \quad w_2 \vee w_3$$
$$\overline{u_1} \vee \overline{u_2} \vee z_1 \vee x_2 \vee z_3$$
$$\overline{u_1} \vee \overline{u_3} \vee z_1 \vee x_2 \vee z_3$$
$$\vdots$$
$$\overline{u_1} \vee \overline{u_2} \vee \overline{w_1} \vee \overline{w_2} \vee z_1 \vee z_2$$
$$\overline{u_1} \vee \overline{u_2} \vee \overline{w_1} \vee \overline{w_2} \vee z_1 \vee z_3$$
$$\vdots$$
$$\overline{u_2} \vee \overline{u_3} \vee \overline{w_2} \vee \overline{w_3} \vee z_2 \vee z_3$$
$$\vdots$$
$$\overline{u_1} \vee \overline{u_2} \vee \overline{u_3} \vee \overline{w_1} \vee \overline{w_2} \vee \overline{w_3} \vee z_3$$
$$\overline{z_1} \vee \overline{z_2} \quad \overline{z_1} \vee \overline{z_3} \quad \overline{w_2} \vee \overline{w_3}$$

CP^2 Upper Bound

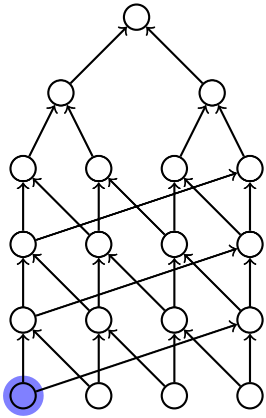


CP² Upper Bound



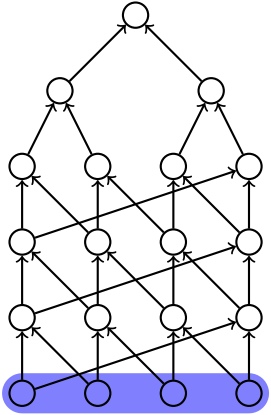
$$x^1_{1,1} + x^2_{1,1} + x^3_{1,1} \geq 2$$

CP^2 Upper Bound



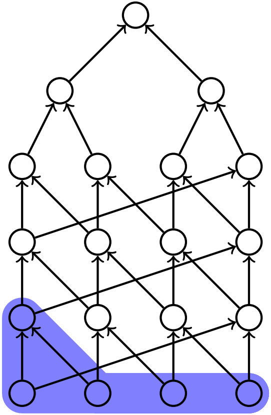
$$2x^1_{1,1} + 2x^2_{1,1} + 2x^3_{1,1} \geq 4$$

CP² Upper Bound



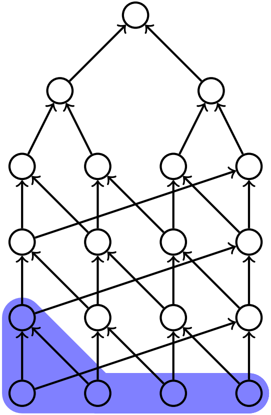
$$2x^1_{1,1} + 2x^2_{1,1} + 2x^3_{1,1} +$$
$$2x^1_{1,2} + 2x^2_{1,2} + 2x^3_{1,2} +$$
$$2x^1_{1,3} + 2x^2_{1,3} + 2x^3_{1,3} +$$
$$2x^1_{1,4} + 2x^2_{1,4} + 2x^3_{1,4} \geq 16$$

CP² Upper Bound



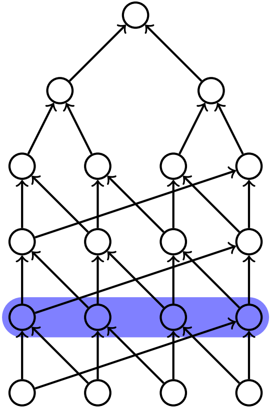
$$
\frac{\begin{array}{l} 2x^1_{1,1} + 2x^2_{1,1} + 2x^3_{1,1} + \\ 2x^1_{1,2} + 2x^2_{1,2} + 2x^3_{1,2} + \\ 2x^1_{1,3} + 2x^2_{1,3} + 2x^3_{1,3} + \\ 2x^1_{1,4} + 2x^2_{1,4} + 2x^3_{1,4} \geq 16 \end{array}}{\begin{array}{c} -x^1_{1,1} - x^2_{1,1} - x^3_{1,1} + \\ -x^1_{1,2} - x^2_{1,2} - x^3_{1,2} + \\ 2x^1_{2,1} + 2x^2_{2,1} + 2x^3_{2,1} \geq 0 \end{array}}
$$

CP^2 Upper Bound



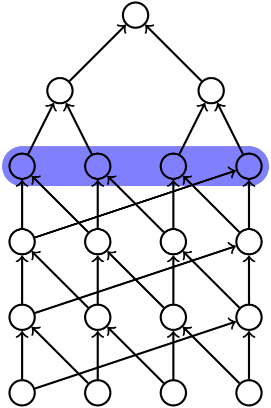
$$
\begin{aligned}
x^1_{1,1} + x^2_{1,1} + x^3_{1,1} + \\
x^1_{1,2} + x^2_{1,2} + x^3_{1,2} + \\
2x^1_{1,3} + 2x^2_{1,3} + 2x^3_{1,3} + \\
2x^1_{1,4} + 2x^2_{1,4} + 2x^3_{1,4} + \\
2x^1_{2,1} + 2x^2_{2,1} + 2x^3_{2,1} \geq 16
\end{aligned}
$$

CP² Upper Bound



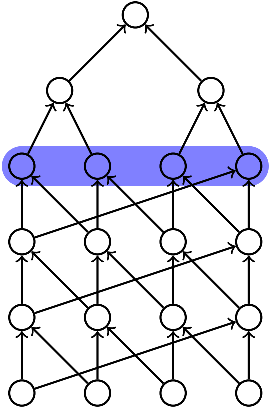
$$2x^1_{2,1} + 2x^2_{2,1} + 2x^3_{2,1} +$$
$$2x^1_{2,2} + 2x^2_{2,2} + 2x^3_{2,2} +$$
$$2x^1_{2,3} + 2x^2_{2,3} + 2x^3_{2,3} +$$
$$2x^1_{2,4} + 2x^2_{2,4} + 2x^3_{2,4} \geq 16$$

CP² Upper Bound



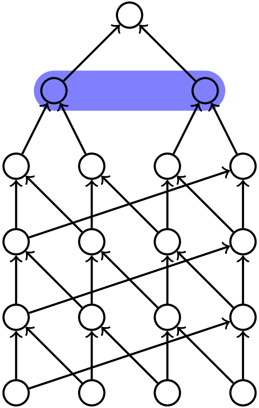
$$2x^1_{4,1} + 2x^2_{4,1} + 2x^3_{4,1} +$$
$$2x^1_{4,2} + 2x^2_{4,2} + 2x^3_{4,2} +$$
$$2x^1_{4,3} + 2x^2_{4,3} + 2x^3_{4,3} +$$
$$2x^1_{4,4} + 2x^2_{4,4} + 2x^3_{4,4} \geq 16$$

CP^2 Upper Bound



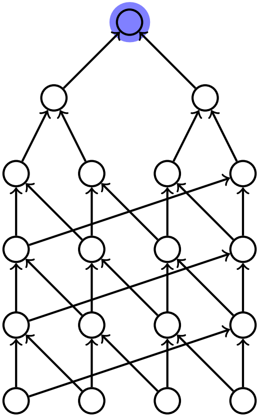
$$
\begin{aligned}
x^1_{4,1} + \; x^2_{4,1} + \; x^3_{4,1} + \\
x^1_{4,2} + \; x^2_{4,2} + \; x^3_{4,2} + \\
x^1_{4,3} + \; x^2_{4,3} + \; x^3_{4,3} + \\
x^1_{4,4} + \; x^2_{4,4} + \; x^3_{4,4} \geq 8
\end{aligned}
$$

CP^2 Upper Bound



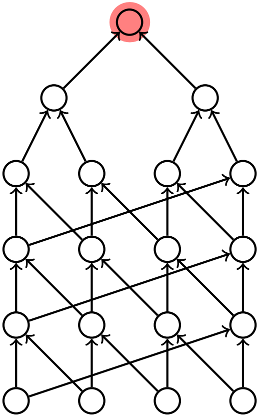
$$2x^1_{5,1} + 2x^2_{5,1} + 2x^3_{5,1} +$$
$$2x^1_{5,2} + 2x^2_{5,2} + 2x^3_{5,2} \geq 8$$

CP² Upper Bound



$$x^1_{6,1} + x^2_{6,1} + x^3_{6,1} \geq 2$$

CP^2 Upper Bound



$$0 \geq 1$$

Resolution Lower Bound

Proof sketch

1. Resolution proof of $\mathrm{Peb}_G[T]$ in line space s.
2. Resolution proof of Peb_G in variable space s.
3. Black-white pebbling of G in s pebbles.
4. G needs $\sqrt{n}$ pebbles.

$F[T]$ to F

Project each configuration $\mathbb{D}$ to a configuration $\mathbb{C}$

$F[T]$ to F

Project each configuration $\mathbb{D}$ to a configuration $\mathbb{C}$

Properties:

- $\mathbb{C}_1, \ldots, \mathbb{C}_t$ is (almost) a resolution refutation of Peb_G
- $\text{VarSp}(\mathbb{C}) \leq \text{Sp}(\mathbb{D})$

$F[T]$ to F

Project each configuration $\mathbb{D}$ to a configuration $\mathbb{C}$

Properties:

- $\mathbb{C}_1, \ldots, \mathbb{C}_t$ is (almost) a resolution refutation of Peb_G
- $\mathrm{VarSp}(\mathbb{C}) \leq \mathrm{Sp}(\mathbb{D})$

Let $\mathbb{B} = \text{Peb}_G[T] \setminus \text{Peb}_G[\geq]$.
$C \in \mathbb{C}$ if

- $\mathbb{D} \cup \mathbb{B}$ implies C; and
- $\mathbb{D} \cup \mathbb{B}$ does not imply any $C' \subset D$.

Recap

	CP*	CP
Length		
Simulates resolution	Y	Y
Separation wrt resolution	Y	Y
Exponential lower bound	Y	Y
Resolution simulates weak division	Y	?
Space		
Simulates resolution	Y	Y
Constant upper bound	?	Y
Superconstant lower bound	CP^k	N
Separation wrt resolution	Y	Y

Recap

	CP^*	CP
Length		
Simulates resolution	Y	Y
Separation wrt resolution	Y	Y
Exponential lower bound	Y	Y
Resolution simulates weak division	Y	?
Space		
Simulates resolution	Y	Y
Constant upper bound	?	Y
Superconstant lower bound	CP^k	N
Separation wrt resolution	Y	Y

Question

Can we separate CP^* *and* CP*?*

Can we Separate Monotone and Real Circuits?

Yes!

f is a k-slice function if $f(x) = \begin{cases} 0 & \mathrm{hw}(x) < k \\ 1 & \mathrm{hw}(x) > k \\ * & \text{otherwise} \end{cases}$

Theorem ([Rosenbloom '97])

Every slice function can be computed with a monotone real circuit of size $\mathrm{O}(n)$

Can we Separate Monotone and Real Circuits?

Yes!

f is a k-slice function if $f(x) = \begin{cases} 0 & \mathrm{hw}(x) < k \\ 1 & \mathrm{hw}(x) > k \\ * & \text{otherwise} \end{cases}$

Theorem ([Rosenbloom '97])

Every slice function can be computed with a monotone real circuit of size $\mathrm{O}(n)$

There are $2^{\binom{n}{n/2}}$ $n/2$-slice functions; therefore most slice functions require exponential boolean circuits.

Can we Separate Monotone and Real Circuits?

Yes!

f is a k-slice function if $f(x) = \begin{cases} 0 & \mathrm{hw}(x) < k \\ 1 & \mathrm{hw}(x) > k \\ * & \text{otherwise} \end{cases}$

Theorem ([Rosenbloom '97])

Every slice function can be computed with a monotone real circuit of size $O(n)$

There are $2^{\binom{n}{n/2}}$ $n/2$-slice functions; therefore most slice functions require exponential boolean circuits.

But this is not explicit…

Can we Explicitly Separate Monotone and Real Circuits?

Want a more delicate argument than counting.

Can we Explicitly Separate Monotone and Real Circuits?

Want a more delicate argument than counting.

Theorem ([Valiant '86])

If a slice function f has a boolean circuit of size m, then f has a monotone boolean circuit of size $m + \mathrm{O}(n \log^2 n)$.

An $\omega(n \log^2 n)$ lower bound would yield superlinear circuit lower bounds.

Can we Explicitly Separate Monotone and Real Circuits?

Want a more delicate argument than counting.

Theorem ([Valiant '86])

If a slice function f has a boolean circuit of size m, then f has a monotone boolean circuit of size $m + \mathrm{O}(n \log^2 n)$.

An $\omega(n \log^2 n)$ lower bound would yield superlinear circuit lower bounds.

Maybe too ambitious…

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.
- Protocol of communication $c \equiv$ monotone circuit of size 2^c

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.
- (Real) protocol of communication $c \equiv$ (real) monotone circuit of size 2^c

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.
- (Real) protocol of communication $c \equiv$ (real) monotone circuit of size 2^c

Real communication strictly more powerful than deterministic:
Can solve EQ with $\mathrm{O}(1)$ real communication, but need $\Omega(n)$ deterministic.

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.
- (Real) protocol of communication $c \equiv$ (real) monotone circuit of size 2^c

Real communication strictly more powerful than deterministic:
Can solve EQ with $\mathrm{O}(1)$ real communication, but need $\Omega(n)$ deterministic.

Natural candidate: threshold. $f(x) = [\![\mathrm{hw}(x) \geq n/2]\!]$.
It is a slice function and has monotone circuits of size $\mathrm{O}(n \log n)$.

Communication Complexity

Karchmer–Wigderson game

- Alice gets $x \in f^{-1}(0)$, Bob gets $y \in f^{-1}(1)$.
- Compute i such that $x_i = 0$ and $y_i = 1$.
- (Real) protocol of communication $c \equiv$ (real) monotone circuit of size 2^c

Real communication strictly more powerful than deterministic:
Can solve EQ with $\mathrm{O}(1)$ real communication, but need $\Omega(n)$ deterministic.

Natural candidate: threshold. $f(x) = [\![\mathrm{hw}(x) \geq n/2]\!]$.
It is a slice function and has monotone circuits of size $\mathrm{O}(n \log n)$.

Another candidate: composition. $f \circ g$, where g is threshold.
Does it inherit the query complexity properties of f?

Take Home

Size of coefficients in cutting planes poorly understood

Thanks!