How Limited Interaction Hinders Real Communication (and What it Means for Proof and Circuit Complexity)

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August 12, University of Toronto, Canada

The SAT Problem

SAT solvers

- Very fast for industrial instances
- Scaling up to millions of variables
- But SAT is NP-complete!

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Proof complexity

- Examples of hard formulas
- Only theoretical tool so far
- Also easy formulas but hard in practice
  Why?

Proof Systems

Resolution

- Logic reasoning
- Most current SAT solvers
- Very well understood

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Cutting planes

- Pseudoboolean reasoning
- Experimental solvers
- Not well understood

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Cutting planes

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- Not well understood

Sums of squares

- Semidefinite programming
- Not used for SAT yet
- Not well understood

Cutting Planes

Work with inequalities

$$x \vee \overline{y} \quad \rightarrow \quad x + (1 - y) \geq 1 \quad \rightarrow \quad x - y \geq 0$$

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Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (a_i + b_i) x_i \geq a + b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

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Goal: derive $0 \geq 1$

Complexity Measures

Size # bits in proof

- Size $2^{O(N)}$ always possible.

Length # lines in proof

- Worst case $2^{\Omega(N^{\epsilon})}$. [Pudlák '97]

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Total space max # bits in memory at the same time

- Space $O(N^2)$ always possible; worst case $\Omega(N)$.

Line space max # lines in memory at the same time

- Space 5 always possible. [Galesi, Pudlák, Thapen '15]

Trade-offs

Question

Assume F has a proof in length L and another proof in space s.
Is there a proof in length $O(L)$ and space $O(s)$?

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Assume F has a proof in length L and another proof in space s.
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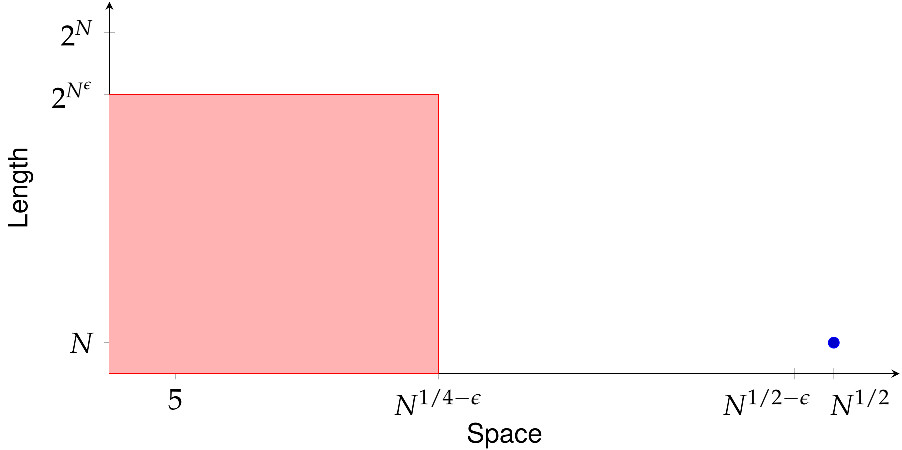
No

Previously studied for resolution and polynomial calculus

[Ben Sasson, Nordström '11] [Beame, Beck, Impagliazzo '12] [Beck, Nordström, Tang '13]

Trade-offs

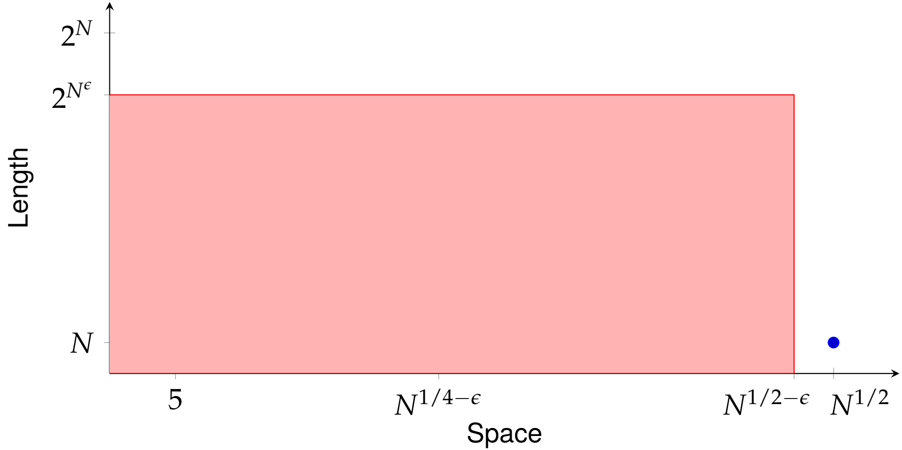


[Huynh, Nordström ’12]

Can do length $\mathrm{O}(N)$, space $N^{1/2}$.

But space $N^{1/4-\epsilon}$ requires size $\exp(N^{\epsilon-\mathrm{o}(1)})$.

Trade-offs

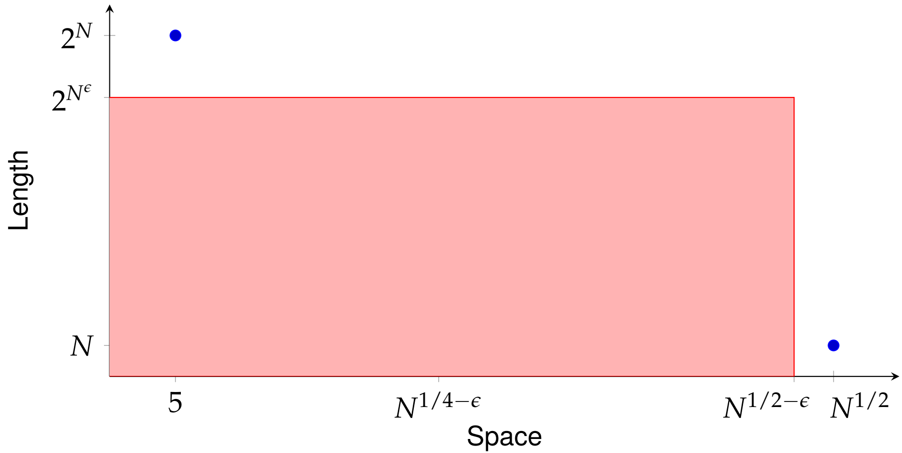


[Göös, Pitassi '14]

Can do length $N^{1+o(1)}$, space $N^{1/2+o(1)}$.

But space $N^{1/2-\epsilon}$ requires size $\exp(N^{\epsilon-o(1)})$.

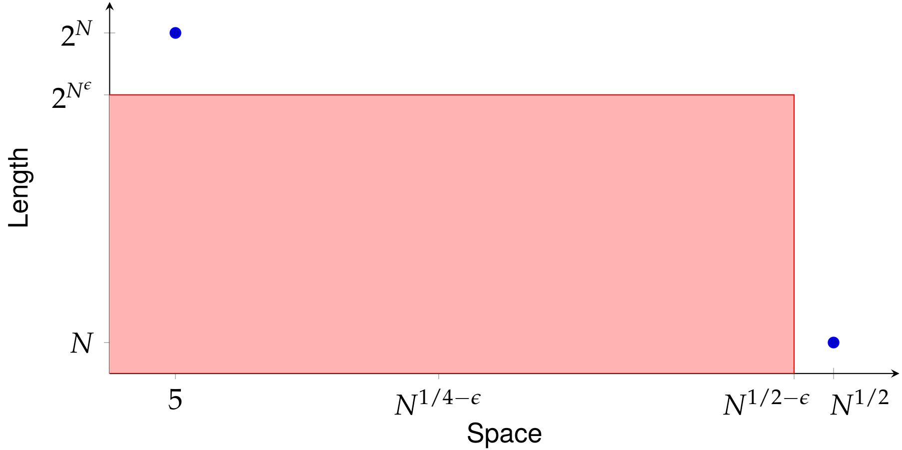
Trade-offs



[Galesi, Pudlák, Thapen '15]

Can do length 2^N, space 5.

Trade-offs



[Galesi, Pudlák, Thapen ’15]

Can do length 2^N, space 5.

But exponential coefficients and quadratic total space.

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Assume F has a proof in small total space with polynomial coefficients. Are there still trade-offs?

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Assume F has a proof in small total space with polynomial coefficients.
Are there still trade-offs?

Cannot answer with previous techniques (provably)

This talk:

Yes

Main Result

Theorem

There is a family of 6-CNF formulas with

- *short proofs: size $\mathrm{O}(N)$, total space $\mathrm{O}(N^{2/5})$;*

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- Upper bounds with constant coefficients, counting all bits.
- Lower bound with unbounded coefficients, only counting lines.
- Lower bound for semantic cutting planes.

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- Lower bound with unbounded coefficients, only counting lines.
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- Holds for resolution and polynomial calculus proof systems.

Spin-off

Exponential separation of the monotone-AC hierarchy

Theorem

There is a monotone Boolean function with

- *small monotone circuits: size* $\mathrm{O}(n)$*, depth* $\log^i(n)$*, fan-in* $n^{4/5}$
- *but monotone circuits of depth* $\mathrm{O}(\log^{i-1} n)$ *require size* $\exp(\Omega(n^\epsilon))$*.*

Superpolynomial separation known [Raz, McKenzie '97]

Devious Plan

Assume refutation in length L and space s

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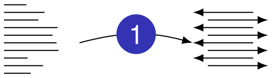
$\downarrow$

1. Communication protocol for falsified clause search problem



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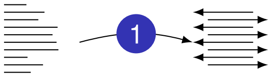
1. Communication protocol for Search(F)



Devious Plan

Assume refutation in length L and space s

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1. Communication protocol for Search(F)

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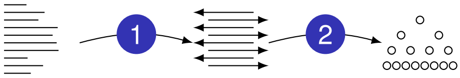

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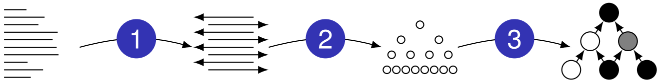
3. Strategy for Dymond–Tompa pebble game



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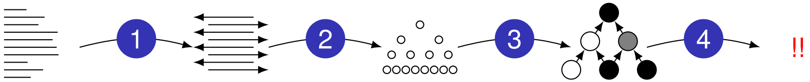
4. Construct graph with trade-offs



Devious Plan 1: Proof $\to$ Protocol

Refutation in length L, space $s \to$
Protocol for Search(F) in $\log L$ rounds, communication $s \log L$

- Inspired by [Beame, Pitassi, Segerlind '05] [Beame, Huynh, Pitassi '10], explicit in [Huynh, Nordström '12].
- Key twists:
  - Real communication model
  - Measure number of rounds

Real Communication

Introduced in [Krajíček '98] to study cutting planes

- Compare real numbers at cost 1

Alice

$\geq$

Referee

Bob

Real Communication

Introduced in [Krajíček '98] to study cutting planes

- Compare real numbers at cost 1

Alice $\xrightarrow{-10^6,\, e^{\pi}}$ $\boxed{\geq}$ Referee $\xleftarrow{8,\, \pi^{e}}$ Bob

Real Communication

Introduced in [Krajíček '98] to study cutting planes

- Compare real numbers at cost 1

$\xrightarrow{-10^6, e^\pi}$ $\boxed{\geq}$ $\xleftarrow{8, \pi^e}$

Alice $\xleftarrow{0, 1}$ Referee $\xrightarrow{0, 1}$ Bob

Real Communication

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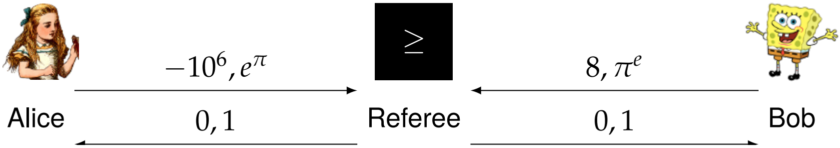

- Simulates deterministic communication (Alice sends m, Bob sends $1/2$)
- Stronger than deterministic communication (EQ)

Devious Plan 1: Proof $\to$ Protocol

Falsified clause search on CNF $F(x, y)$

- Alice $\leftarrow$ assignment to x variables
- Bob $\leftarrow$ assignment to y variables
- Task: Find falsified clause

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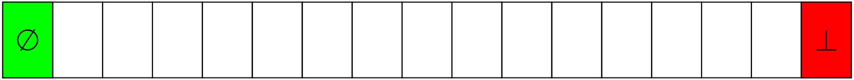

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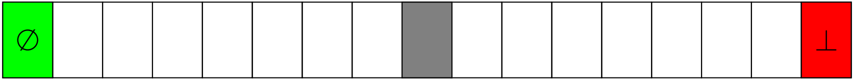

- Alice evaluates $\sum a_i x_i - a$ in s inequalities
- Bob evaluates $-\sum a_i y_i$ in s inequalities
- $\alpha(\mathbb{C}) = 1$ iff Referee answers $111\ldots1$

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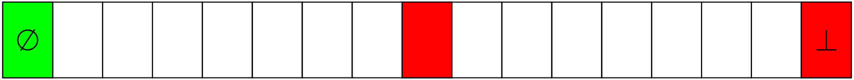

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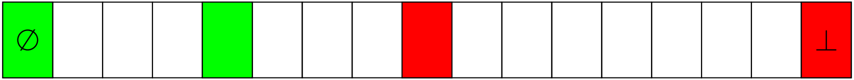

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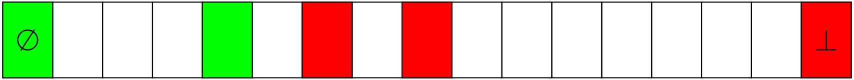

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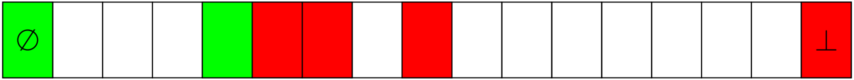

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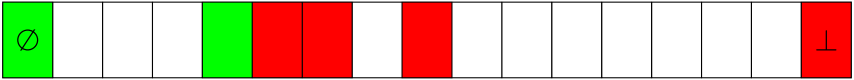
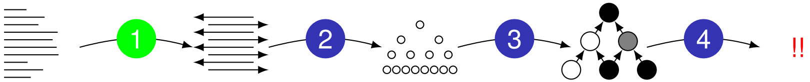


- $\alpha(\mathbb{C}) = 1 \quad \alpha(\mathbb{C} \cup \{A\}) = 0 \quad \Rightarrow \quad \alpha(A) = 0$
- $\log L$ rounds, communication $s \log L$

Devious Plan

Assume refutation in length L and space s

1. Communication protocol for Search(F) in $\log L$ rounds and communication $s \log L$
2. Parallel decision tree for Search(F)
3. Strategy for Dymond–Tompa pebble game
4. Construct graph with trade-offs



Devious Plan 2: Protocol $\to$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\to$
Parallel decision tree for S of depth r, c queries

Lifted Problem

- Function $f(z_1, \ldots, z_n)$
- Alice $\leftarrow$ n indices $x_1, \ldots, x_n$
- Bob $\leftarrow$ n arrays $y_1, \ldots, y_n$

$z_1 = y_1[5] = 1$ $\quad$ 5 $\quad$

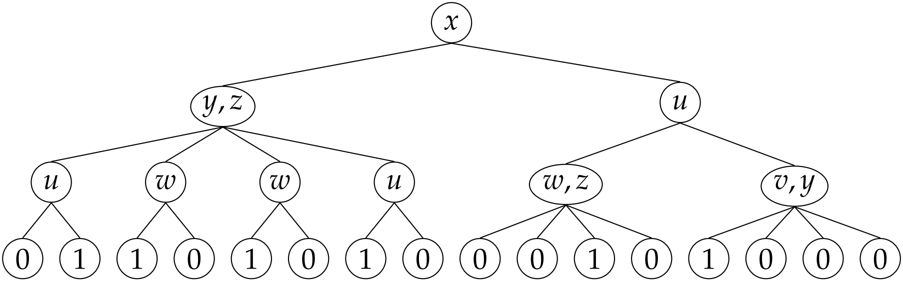
0	0	1	0	0	1	1	1

x_1 $\quad$ y_1

- Lifted function $\text{Lift}(f)(x, y) = f(y_1[x_1], \ldots, y_n[x_n])$

Parallel Decision Trees

Decision tree with many queries per node [Valiant '75]



Depth Longest branch

Queries # queries in a branch

Devious Plan 2: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
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Devious Plan 2: Protocol $\leftarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\leftarrow$
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Communication

Decision tree

Query $\{z_3, z_{28}\}$

Devious Plan 2: Protocol $\leftarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\leftarrow$
Parallel decision tree for S of depth r, c queries

Communication

Alice sends x_3, x_{28}
Bob sends $y_3[x_3], y_{28}[x_{28}]$

Decision tree

Query $\{z_3, z_{28}\}$

Devious Plan 2: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
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Communication

Alice sends $x_1 + x_2 + \cdots + x_n$

Decision tree

Devious Plan ❷: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

Communication

Alice sends $x_1 + x_2 + \cdots + x_n$

Decision tree

???

Devious Plan 2: Protocol $\rightarrow$ Decision Tree

Protocol for $\mathrm{Lift}(S)$ in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

- Main technical result (Simulation Theorem)
  - Technique from [Raz, McKenzie '97]
  - Adapted to real communication in [Bonet, Esteban, Galesi, Johannsen '98]
  - Connection to decision trees made explicit in [Göös, Pitassi, Watson '15]
- Our contribution
  - Introduce rounds
  - Adapt to real communication preserving rounds

Devious Plan

Assume refutation of lifted formula in length L and space s

1. Communication protocol for Lift$($Search$(F))$ in $\log L$ rounds and communication $s \log L$
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3. Strategy for Dymond–Tompa pebble game

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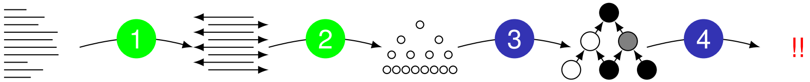


Devious Plan ❸: Decision Tree $\rightarrow$ Dymond–Tompa

Parallel decision tree for Search(Peb_G) of depth r, c queries $\leftrightarrow$ Dymond–Tompa pebble game strategy for r rounds, c pebbles

Pebbling Formulas

- Sources are true

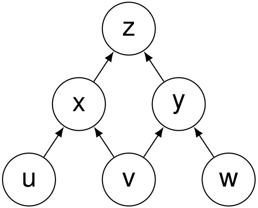
$$u$$
$$v$$
$$w$$

- Truth propagates

$$(u \wedge v) \rightarrow x$$
$$(v \wedge w) \rightarrow y$$
$$(x \wedge y) \rightarrow z$$

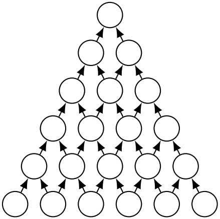
- Sink is false

$$\overline{z}$$



Dymond–Tompa Game

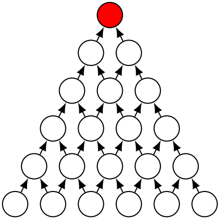
2-player pebble game on a DAG [Dymond, Dompa '85]



Dymond–Tompa Game

2-player pebble game on a DAG [Dymond, Dompa ’85]

- Start with a challenged pebble on the sink



Rounds	0
Pebbles	1

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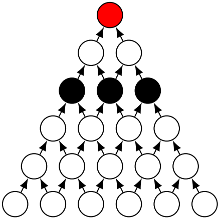
- Start with a challenged pebble on the sink
- Each round:
  - Pebbler adds some pebbles



Rounds 1

Pebbles 4

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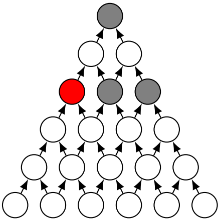

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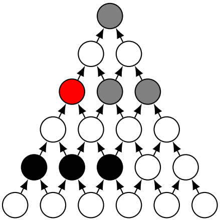

Rounds 2

Pebbles 7

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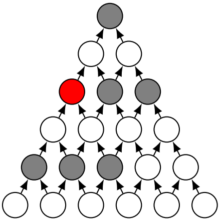

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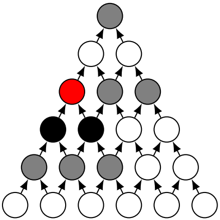


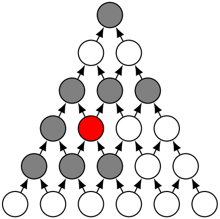
Rounds 3

Pebbles 9

Dymond–Tompa Game

2-player pebble game on a DAG [Dymond, Dompa '85]

- Start with a challenged pebble on the sink
- Each round:
  - Pebbler adds some pebbles
  - Challenger may challenge one new pebble
- Ends when challenged pebble is surrounded



Rounds 3

Pebbles 9

Devious Plan 3: Decision Tree $\to$ Dymond–Tompa

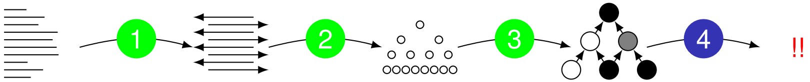
Parallel decision tree for $\mathrm{Search}(\mathrm{Peb}_G)$ of depth r, c queries $\leftrightarrow$ Dymond–Tompa pebble game strategy for r rounds, c pebbles

- Done in [Chan '13]
- Tweak to preserve rounds

Devious Plan

Assume refutation of lifted pebbling formula in length L and space s

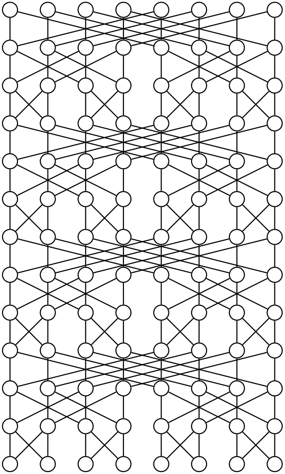
1. Communication protocol for Lift(Search(F)) in $\log L$ rounds and communication $s \log L$
2. Parallel decision tree for Search(F) of depth $\log L$ and $s \log L$ queries
3. Strategy for Dymond–Tompa pebble game for $\log L$ rounds and $s \log L$ pebbles [Chan '13]
4. Construct graph with trade-offs



Devious Plan 4: Trade-off for Dymond–Tompa

Graph where r-round DT game needs $n/4$ pebbles

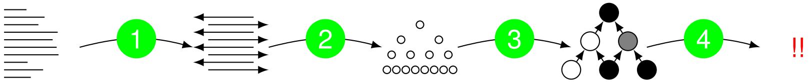
- Stack of $r+1$ butterfly graphs
- Can do $2r\log n$ pebbles in $r\log n$ rounds
- Or $n\log(r\log n)$ pebbles in $\log(r\log n)$ rounds



Devious Plan

Assume refutation of lifted pebbling formula in length L and space s

1. Communication protocol for Lift$(\text{Search}(F))$ in $\log L$ rounds and communication $s \log L$
2. Parallel decision tree for Search(F) of depth $\log L$ and $s \log L$ queries
3. Strategy for Dymond–Tompa pebble game for $\log L$ rounds and $s \log L$ pebbles
4. Construct graph where such strategy does not exist



Take Home

Remarks

- Strong size-space trade-offs for cutting planes
- Hold for resolution, polynomial calculus, cutting planes
- Key to measure rounds

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- Smaller lift size
- Stronger models of communication

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Thanks!