$\boxed{A} \xleftrightarrow{\text{msg}} \boxed{B}$

x $\qquad\qquad$ y

compute $f(x,y)$.

e.g. $x+y$ even?

$A \xrightarrow{z = x \bmod 2} B$

$\xleftarrow[\text{answer}]{}$ $\quad$ 2 bits

e.g. $6 \mid x+y$?

$A \xrightarrow{x \bmod 2} B$

$\xleftarrow{}$

no, done / maybe, cont

$A \xrightarrow{x \bmod 3} B$

$\xleftarrow[\text{answer}]{}$ $\quad$ max(2 bits, 5 bits)

e.g. $x = y$? need n bits.

enter randomness

$\boxed{A} \longleftrightarrow \boxed{B}$

$x, \underline{r}$ $\qquad\qquad$ $y, \underline{r}$

answer correct w/ prob 3/4 over r.

can solve EQ like this

→ choose random prime among n first

→ A sends $x \bmod p$

→ B says yes iff $x = y \bmod p$.

now can solve other problems

e.g. greater-than $x \geq y$?

e.g. small-set disjointness. $\quad$ x set of k elements

y — " —

$x \cap y = \emptyset$?

but... all of these can be solved just with EQ.

what do I own with this?

$\boxed{A} \underset{=?}{\overset{f(x)}{\rightleftarrows}} \boxed{=} \underset{=?}{\overset{g(y)}{\leftrightarrows}} \boxed{B}$

$x\ \cancel{y}$ $\qquad\qquad$ $\cancel{x}\ y$

Q. for every f, is P^{EQ} cost $\simeq$ BPP cost?

// Known false for partial functions.

e.g. $\mathrm{Maj}(x \oplus y)$, promise $x \oplus y$ has either $n/3$ 0s or $n/3$ 1s //

Q. for every total f, is P^{EQ} cost $\simeq$ BPP cost?

we prove: no.

function:

input $x_1 \dots x_t$ $\quad$ t numbers in $[-N, N]$.

$y_1 \dots y_t$

def $\mathrm{IIP}(x,y)$: $\langle x, y\rangle = 0$?

th. IIP has BPP cost $O(t \log n)$

but P^{EQ} – $\Omega(n)$.

proof

But what if we had IIP as the oracle?

$\boxed{A} \rightleftarrows \boxed{\mathrm{IIP}} \leftrightarrows \boxed{B}$

th $\forall t\ \exists t'$ st P^{IIP_t} cost of $\mathrm{IIP}_{t'}$ is $\Omega(n)$

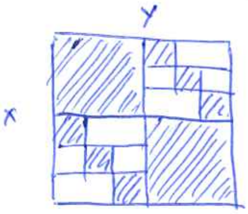
so... $P^{EQ} \subsetneq P^{\mathrm{IIP}_{t_1}} \subsetneq P^{\mathrm{IIP}_{t_2}} \dots \subset \mathrm{BPP}$.

Proof sketch.

Upper bound: like EQ. sample large prime.

Lower bound. look at comm matrix.

recall $G(x,y)$.

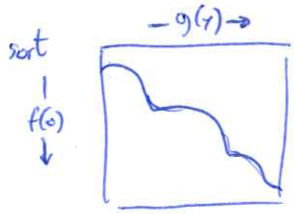


each bit splits matrix into 2 rects.

after c bits have $\le 2^c$ rects.

show EQ requires $2^{\Omega(n)}$ rects.

with P^{EQ}. (actually P^{GT}) (can solve EQ in 2 calls)

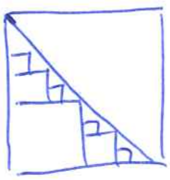


each bit splits into 2 triangles.

problem: ~~each call may use different order~~

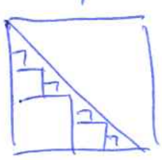
intersections of triangles are not triangles.

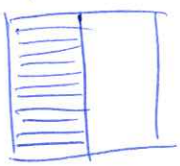
Let us split a triangle into rectangles.



problem: 1 call already gives 2^n rectangles.

but... many of these are large. can we exploit it?





$\frac{1}{2}$ perimeter $= 1\cdot\left(\frac{1}{2}+\frac{1}{2}\right) + 2\cdot\left(\frac{1}{2}\right) + \dots + 2^n\cdot\frac{1}{2^n}$

$2^n\cdot\left(\frac{1}{2}+\frac{1}{2^n}\right) = 2^{n-1}$

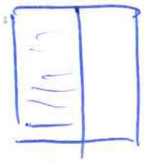
$1 + 1 + \dots = n$

much better!

slightly better measure: η-area $(a \times b) = (ab)^\eta$.

$1\cdot\left(\frac{1}{4}\right)^\eta + 2\cdot\left(\frac{1}{16}\right)^\eta + \dots \le \frac{1}{2}\sum 2^{2\eta-1} \to q$

if $\eta > 1/2$



$2^n\cdot\frac{1}{2^{\eta n}} \approx 2^{n-\eta n} = 2^{n(1-\eta)}$.

Claim: each call increases η-area by q.

after c calls total η-area is q^c.

Lemma: for each η $\exists t$ st IIP_t has η-area $2^{\Omega(n)}$.