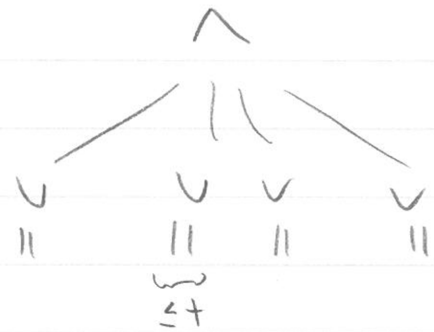
My view of switchig.

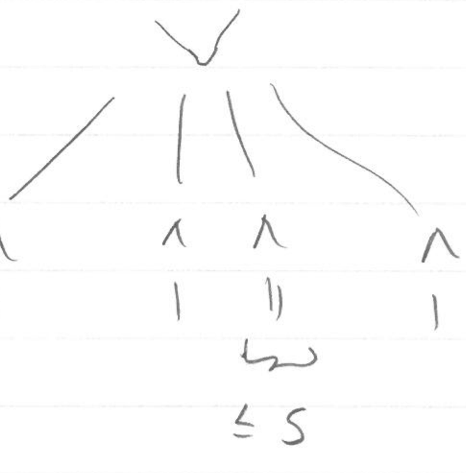
Switching is taking a depth two circuit

$F =$



and by assigning values to some of the variables we make it possible to make $F\restriction_\rho$ a decision tree of depth $\leq s$ and in particular the other type of depth 2 circuit.

$F\restriction_\rho =$



Useful for proving lower bounds for circuits and proof complexity

In proof complexity it makes it possible to simplify k-evaluations that I think of as approximations of subformulas set up in a way to preserve local derivations.

We need to balance two properties.

ρ does not make the problem we are working on easy

ρ does enable us to switch.

Three set-ups to consider

1. Proving PHP

2. Computing parity.

3. Proving Tseitin tautology

Recall Tseitin

Graph:
Variables ~ edges
$\sum_{u \sim w} x_{uw} = \alpha_u \mod 2$

Contradiction if $\sum \alpha_u = 1 \mod 2$

Natural spaces of restrictions

1) Match $n-\ell$ pigeons to $n-\ell$ holes

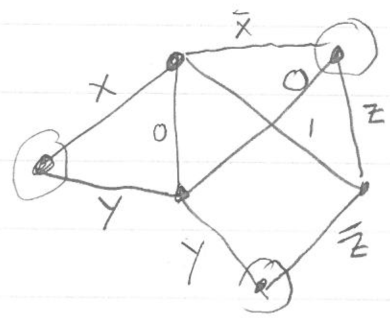
Clean $\ell+1, \ell$ problem remains.

2) Any assignment to $n-\ell$ variables

R_p Keep each variable with prob p
Parity of ℓ variables remain.

3. Find induced graph

new nodes ~ some old nodes
new edges ~ paths between chosen node.



Old variables ~ New variable
Negated -||-
Constants

More general items than standard restriction, but not difficult.

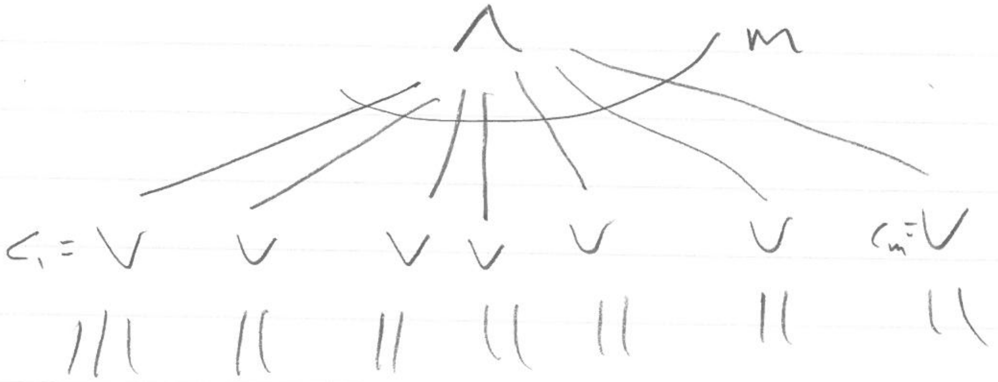
Switching lemma parameters

How do we bound input and output fanins.

When can we hope it is true/provable.

What does this give for maximal depth we can prove something for?

Test case to get a feeling for parameters.



disjoint of size exactly t

For intuition, what happens when we have no ρ?

We get decision tree of size $\sim tm$, nothing interesting happens.

Why do we do better now?

$C_i \to 1$ Neutral

$C_i \to 0$ Great

$C_i \to$ Remains undetermined Bad.

A single 0 kills the entire formula.

Too many clauses ⇒ Likely 0

Too few clauses ⇒ Few undermined.

Key parameter

$$\frac{\Pr[C_i \text{ undetermed}]}{\Pr[C_i\restriction_\rho \equiv 0]}$$

If this is ≤ 1
probability getting s undetermined without any 0 is 2^{-s}.

And exponential decay is s is what we hope for.

Now let us see what this number is

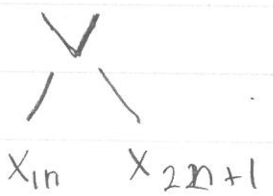
$$R_p \quad \frac{\left(\frac{1+p}{2}\right)^t - \left(\frac{1-p}{2}\right)^t}{\left(\frac{1-p}{2}\right)^t} \approx pt$$

if $pt \ll 1$

The other distributions even this simple calculation is non obvious.

PHP

Take even $t = 2$ and



False $\approx 1 - \frac{2\ell^2}{n^2} - \frac{2}{n}$

Undetermined $\approx \frac{2\ell^2}{n^2}$

True $\approx \frac{2}{n}$

Better have $\frac{\ell^2}{n^2} < n \quad \ell < \sqrt{n}$

to have any hope of switching

Tseitin

Let us also consider $t=2$.

0 and 1 are symmetric

Key ratio

$$\frac{\text{Variables set to new variables}}{\text{Variables set to } 0/1}$$

We want this $\leq 1/t$ to have hope.

n nodes $\longrightarrow$ n' nodes

$3n'/2$ paths of length ?

Short paths are important.

Expander graphs $\log n$
Grid graphs $\sqrt{n/n'}$

Both ok at this level

Not that for PHP

we go from n to at most $\sqrt{n}$

Repeating d times we have

$$n^{2^{-d}} = 2^{2^{-d} \cdot \log n}$$

nodes left $d > \log\log n$ is a problem.

please consult 9.5 + 9.6

Others we might have

$s = t \sim O(\log n)$.

$p \sim (\log n)^{-1}$

$n'/n \sim (\log n)^{-2}$

both giving $\frac{n}{(\log n)^{O(d)}} = 2^{\log n - O(d \log\log n)}$

might have : $d = \Omega(\log n / \log\log n)$

(9.5)

Proving lower bounds for

$PHP_n^{n+\log^a n}$

with restrictions. We need to keep $n/\log^b n$ variables after a restriction.

Restrict pigeons to use edges in a degree d graph.

Keep $n/\log^b n$ variables

$d \longrightarrow d/\log^b n$

Probability of keeping a variable *

$\sim \frac{1}{(\log^b n)^2}$

Probability of being $\underline{1}$ is

$\frac{1}{d}$

(9.6)

If d is between

$(\log n)^b$ and $(\log n)^{2b}$

there is hope.

As $\frac{d}{\log n} < \sqrt{d}$

we still have at most

$(\log\log n)$ levels her

this was a ballpark argument for independent clauses.

when can we prove it for general dependant clauses?

Really nice spaces of random restrictions like R_p.
Induction.

Fairly nice spaces of random restrictions like
"keep exactly ℓ variables"
Labeling argument

Let us do the simplest case

A set Δ of restrictions is downward closed if $\rho \in \Delta$ and $\rho'(x_i) = \rho(x_i)$ whenever $\rho(x_i) \in \{0,1\}$ then $\rho' \in \Delta$.

Changing the value from $*$ to 0/1 on any input does not make us leave Δ.

Ex: All ρ $F\restriction_\rho \equiv 1$

All ρ that make F into a decision tree of depth $\le s$

All ρ that hands out at most pn $*$'s.

Lemma: Suppose f is computed by depth 2 circuit of bottom fanin $\le t$, Δ downward closed

$$\Pr[\, \mathrm{depth}(f\restriction_\rho) \geq s \mid \rho \in \Delta \,] \leq (6pt)^s$$

Proof: $f = \bigwedge_{i=1}^{m} C_i$

induction over m.

Either $C_1\restriction_\rho \equiv 1$ or not

$$\Pr[\, \mathrm{depth}(f\restriction_\rho) \geq s \mid \rho \in \Delta \wedge C_1\restriction_\rho \equiv 1 \,]$$

$$\Pr[\qquad \not\equiv]$$

First is small by induction.

For the second let us consider C_1. There must exist some non-empty set Y of the variables in C_1 that is given values $*$.

Let τ be some values given to these variable.

$$\sum_{\tau} \sum_{Y \neq \emptyset} \Pr[\rho(Y) = *,\ \rho(C_1 \setminus Y) \text{ constants} \mid \rho \in \Delta]$$

$$\cdot \Pr[\text{depth}(f\lceil_{\rho\tau}) \geq s - r \mid \rho(Y) = *\ \rho(C_1 \setminus Y) = \cdot\ \rho \in \Delta]$$

Lemma: $\Pr[\rho(Y) = * \mid \rho \in \Delta] \leq \left(\frac{2p}{1+p}\right)^{|Y|}$

Proof: Take any ρ contributing to the probability and change to non-$*$ on Y.

From ρ we get restrictions

of total probability mass

$$\sum_{Z \subseteq Y} \left(\frac{1-p}{2p}\right)^{|Z|} \Pr[\rho] =$$

$$\left(\frac{1+p}{2p}\right)^{|Y|} \cdot \Pr[\rho]$$

□.

Now we get total

$$\sum_{\substack{Y \neq \emptyset \\ |Y|=r}} 2^r \cdot \left(\frac{2p}{1+p}\right)^r (6pt)^{s-r} =$$

$$(6pt)^s \sum_Y \left(\frac{4p}{(1+p)6pt}\right)^r =$$

$$(6pt)^s \left(\left(1 + \frac{4}{6t}\right)^t - 1\right) \approx$$

$$(6pt)^s \left(e^{2/3} - 1\right) \leq (6pt)^s$$

and we are done.

The point is that $\rho \in \Delta$ cannot make ρ be more likely to be $*$ on Y.

Now suppose that we instead insist on having exactly pn $*$'s.

Then in fact $\rho \in \Delta$ can make $*$'s more likely.

$$\rho \in \Delta \iff \rho(x_i) = 0 \quad \text{for } i = 1, \ldots n - pn$$

This is a perfectly downward closed set and forces $*$'s in remaining places.

In PHP we have a similar situation built in

If restriction forces $x_{11}, x_{12} \ldots x_{1n-1}$ to false

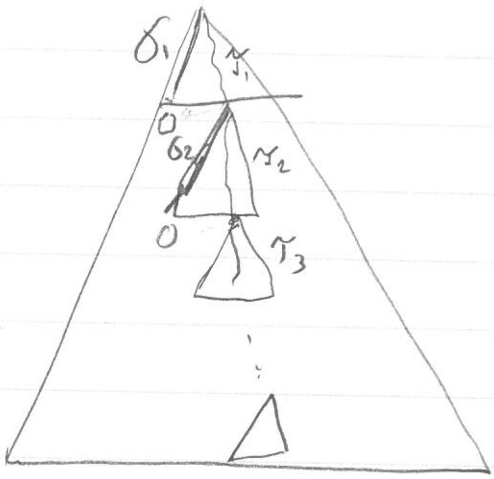
and we know x_{1n} is 0 or $*$ it is likely $*$.

Similarly in Tseitin as we commit to embedding a large graph.

Razborov' labeling.

In simplest case

ρ



$\rho + \sigma$ + values τ.

which variables were given value $*$, (Y)

Which values do they take in the long path in decision tree τ.

Total number of $\tau \sim 2^s$

Total number of y's $(2r)^s$

We map from restrictions with pn $*$'s to $pn-s$ $*$.

This gives a factor

$$\frac{R_{pn-s}}{R_{pn}} = 2^s \cdot \frac{\binom{n}{pn-s}}{\binom{n}{pn}} \sim 2^s \cdot p^s$$

and we are done.

Now what are the complications with PRST?

In restrictions we have see so far, there is rather small effects of setting a variable.

Tseitin is a mess. A variable is set to a value 0/1 does cause a lot of problems and we have much worse problems with conditioning.

In the R_p case we were able to ignore clauses forced to one.

Suppose instead we each time we are confronted with clause make sure that all variables are set.

I.e even if C_1 is forced true we ask any $*$'s that appear in the clause.

Weaker lemma:

$$\Pr[\ \mathrm{depth}(F\restriction_p) \geq s] \leq c^s$$

where $c =$

Look at C_1.

Probability $1 - (1-p)^t \approx tp$ of something bad "putting something in decision tree".

Probability $(1-p)^t \cdot 2^{-t} \approx 2^{-t}$ something good. Formula $\Rightarrow 0$

If $pt < 2^{-t}$ there is hope.

calculation

Y set of $*$ variables

If non-zero we have values J as in standard proof

$$\sum_Y |Y|^p (1-p)^{t-|Y|} \, 2^{|Y|} \cdot c^{s-t}$$

$$= \left(1-p+\frac{2p}{c}\right)^t \cdot c^s$$

but we are overcounting when

$Y = \emptyset$ with probability 2^{-t} we should have 0 and not c^s. Need

$$\left(1-p+\frac{2p}{c}\right)^t - 1 \leq 2^{-t} \cdot (1-p)^t$$

$$t\left(\frac{2p}{c}-p\right)^{22} \leq 2^{-t} \qquad c \geqslant 2p \cdot t \cdot 2^{-t}.$$

ignoring some details.

Ignoring some points

$p = 2^{-2t}$

$t = s$

We have error probability

$\sim \left(2^{-t}\right)^s = 2^{-t^2}$

we need to have some variables left after d rounds

$n \cdot \left(2^{-2t}\right)^d > 1 \qquad t = \frac{\log n}{d}$

we get bounds $2^{-\Omega\left(\left(\frac{\log n}{d}\right)^2\right)} =$

$n^{-\frac{\log n}{d^2}}$ and $d < \sqrt{\log n}$

Now we need to do also this with Tseitin restrictions.

It is a mess but believable.

As there are no conditional statement we might try to prove a switching lemma without labeling.

There has been some bugs in the proof but I am rather convinced that this can be fixed.

The real challenge is coming up with a type of graph and restrictions where you can pull off a switching lemma.

Probably a labelling argument as Razborov.