Lecture 14: "PHP is hard for bd Frege" - part II -

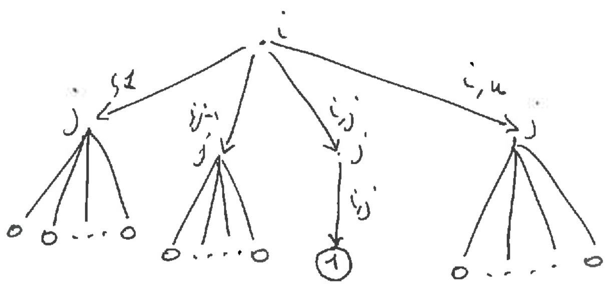
recap:

Last lecture we introduced the notion of k-evaluation and proved (modulo some lemmas we are going to prove now) the following theorem.

Thm: Let $\mathcal{F}$ be a Frege system over $\{\vee, \neg\}$ and let $d > 3$. For sufficiently large n, every depth d proof of $\neg OFPHP_n^{n+1}$ in $\mathcal{F}$ has size $\geq 2^{n^{\delta^d}}$ for $0 < \delta < \left(\frac{1}{5}\right)$.

There are left to prove 3 lemmas: Lemma 1 which informally says that given a k-evaluation v, $v(\neg OFPHP_n^{n+1})$ has all leaves 0, Lemma 2 that shows how along proofs in $\mathcal{F}$ every line F is such that $v(F)$ has all leaves 1. Lemma 3 (next lecture) will show how to build a k-evaluation.

Remark: To simplify some definitions and notations later on it is convenient to consider CMDT (Complete Matching Decision Trees) where along branches we could re-query some vertex, for instance the tree associated a k-evaluation to x_{ij} is



To prove Lemma 1 and 2 we need some preliminary observations.

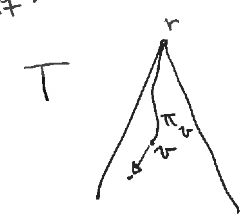
- Some additional notation we will need:

$V = P \cup H \quad \alpha \in M_n \quad V\restriction_\alpha = \{\alpha \in V : \nexists e \in \alpha \ \ e \ni v\}$

CMDT(S) with $S \subseteq V$ is the set of matching decision trees T s.t. the labels of the edges going out from a vertex $i \in T$ must correspond to all possible ways of extending the matching corresponding to the path in T leading to i with an edge (i,j) with $j \in S$.

Obs 1: Let $T \in CMDT_n$ and let $\alpha \in \mathcal{M}_n$ s.t. $|\alpha| + depth(T) \leq n$, then there exists a $\pi \in Br(T)$ s.t. $\alpha \cup \pi \in \mathcal{M}_n$.

proof:



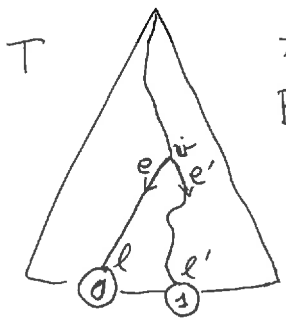
Start with the root r of T and then we construct a path π_v in T s.t. $\alpha \cup \pi_v \in \mathcal{M}_n$.

- $\alpha \cup \pi_r = \alpha \in \mathcal{M}_n$,
- suppose we arrived to a vertex $v \in T$ that is not a leaf and $\alpha \cup \pi_v \in \mathcal{M}_n$, $|\alpha \cup \pi_v| \leq |\alpha| + |\pi_v| < |\alpha| + depth(T) \leq n$

So there is a way of extending the matching $\alpha \cup \pi_v$ in $K(P,H)$.
Since T is complete there is an extension of the path π_v with such label. ∎

Obs 2: Let $T \in CMDT_n$ let π_ℓ be the matching corresp. to the path leading to the leaf $\ell \in T$. Let $\alpha \supseteq \pi_\ell$, then if ℓ has label 0 then $Disj(T)\restriction_\alpha \equiv 0$, if ℓ has label 1 then $Disj(T)\restriction_\alpha \equiv 1$.

proof: Since $\alpha \supseteq \pi_\ell$, then $\bigwedge_{e \in \pi_\ell} x_e \restriction_\alpha \equiv 1$ so if ℓ has label 1 such term is in $Disj(T)$ and hence $Disj(T)\restriction_\alpha \equiv 1$.
Suppose now ℓ has label 0, we need to show that for any other leaf ℓ' with label 1, $\bigwedge_{e \in \pi_{\ell'}} x_e \restriction_\alpha \equiv 0$.



π_ℓ and $\pi_{\ell'}$ separates in some vertex v
By construction $\tilde{\alpha}(x_{e'}) = 0$ so $\bigwedge_{e \in \pi_{\ell'}} x_e \restriction_\alpha \equiv 0$. ∎

Lemma 1 (restated): Let Γ be a set of formulas containing $\neg OFPHP_n^{n+1}$ and let ν be a k-evaluation of Γ with $k < n-1$, then $\nu(\neg OFPHP_n^{n+1})$ has all leaves 0.

proof: Recall that

$$\neg OFPHP_n^{n+1} = \bigvee_{i\in P} \underbrace{\left(\neg \bigvee_{j\in H} x_{ij}\right)}_{P^i} \vee \bigvee_{\substack{j\in H \\ i\neq i'\in P}} \underbrace{\left(\neg(\neg x_{ij} \vee \neg x_{i'j})\right)}_{H_j^{i,i'}} \vee \bigvee_{j\in H} \underbrace{\left(\neg \bigvee_{i\in P} x_{ij}\right)}_{O_j} \vee \bigvee_{\substack{i\in P \\ j\neq j'\in H}} \underbrace{\left(\neg(\neg x_{ij} \vee \neg x_{ij'})\right)}_{F_{j,j'}^{i}}$$

If $\nu(\neg OFPHP_n^{n+1})$ has some leaf ℓ with label 1, let π_ℓ be the path in $\nu(\neg OFPHP_n^{n+1})$ leading to ℓ. Since $\nu(\neg OFPHP_n^{n+1})$ represents

$$\bigvee_{i\in P} Disj(\nu(P^i)) \vee \bigvee_{\substack{j\in H \\ i\neq i'\in P}} Disj(\nu(H_j^{i,i'})) \vee \bigvee_{j\in H} Disj(\nu(O_j)) \vee \bigvee_{\substack{i\in P \\ j\neq j'\in H}} Disj(\nu(F_{j,j'}^{i})),$$

this means that there must exists some formula φ among the P^i, $H_j^{i,i'}$, O_j and $F_{j,j'}^i$, s.t. $Disj(\nu(\varphi))\restriction_{\pi_\ell} \equiv 1$. π_ℓ then extends some matching $\rho \in Br(\nu(\varphi))$ so by obs 2 there is a leaf in φ with label 1. We show that this is not possible for $\varphi = P^i$. (The cases when $\varphi = H_j^{i,i'}$, $\varphi = O_j$ and $\varphi = F_{j,j'}^i$ are left as exercises.)

We need to show that for any $i \in P$, $\nu(\bigvee_{j\in H} x_{ij})$ has all leaves 1.

$\nu(\bigvee_{j\in H} x_{ij})$ represents

$$\bigvee_{j\in H} Disj(\nu(x_{ij})) \equiv \bigvee_{j\in H} x_{ij}.$$

Let $\sigma \in Br(\nu(\bigvee_{j\in H} x_{ij}))$. If $(i,j)\in\sigma$ for some j then clearly $\bigvee_{j\in H} x_{ij}\restriction_\sigma \equiv 1$ so the leaf corresp to σ must have label 1. If $(i,j)\notin\sigma$ for all j, then since $k<n-1$, by obs 1, there exist a matching $\sigma' \supseteq \sigma$ s.t. $(i,j)\in\sigma'$ for some j so $\bigvee_{j\in H} x_{ij}\restriction_{\sigma'} \equiv 1$ and by obs 2 the leaf corresponding to σ must have label 1.

$\square$

Notice: for $\varphi = H_j^{i,i'}$ and $F_{j,j'}^i$, it is needed to be able to extend twice so it is needed that $k < n-1$ and not just $k \le n-1$ as we did for $\varphi = P^i$.

Lemma 2 (restated from last lecture): Let S be the Shoenfield's system and π any proof in S. Let ν be a k-evaluation of π s.t. $k \leq \frac{n}{7}$ then for every line F in π, $\nu(F)$ has all leaves labeled 1.

proof: By induction on the number of lines in π. We prove that whenever in the proof we apply the rule $\frac{A \vee B \quad \neg B \vee C}{A \vee C}$ and we suppose $\nu(A \vee B)$ $\nu(\neg B \vee C)$ have all leaves 1 then also $\nu(A \vee C)$ has all leaves 1. The cases when we use the rules $\frac{}{A \vee \neg A}$, $\frac{A}{A \vee B}$, $\frac{A \vee A}{A}$, $\frac{A \vee (B \vee C)}{(A \vee B) \vee C}$ are easier and left as exercises.

Let $\sigma \in Br(\nu(A \vee C))$, by obs 1 there exists a $\pi_A \in Br(\nu(A))$ s.t. $\sigma \cup \pi_A \in \mathcal{M}_n$, by obs 1 again there exists a $\pi_B \in Br(\nu(B))$ s.t. $\sigma \cup \pi_A \cup \pi_B \in \mathcal{M}_n$, proceeding in this way we can get a $\pi \in \mathcal{M}_n$ s.t. $\pi \supseteq \sigma$ and there are $\pi_F \in Br(\nu(F))$ s.t. $\pi \supseteq \pi_F$ for each $F \in \{A, B, C, A \vee B, \neg B \vee C\}$. Notice that here to apply several times (6) obs 1 we used that $|\sigma| \leq \frac{n}{7}$ and $k \leq \frac{n}{7}$.

Suppose that $\sigma \in Br(\nu(A \vee C))$ was leading to a leaf with label 0, so by obs 2 $Disj(\nu(A)) \vee Disj(\nu(C))\restriction_\pi \equiv 0$ and in particular $Disj(\nu(A))\restriction_\pi \equiv 0$ and $Disj(\nu(C))\restriction_\pi \equiv 0$. Since $\pi \supseteq \pi_B$ then $Disj(\nu(B))\restriction_\pi$ is either 0 or 1 according whether the leaf in $\nu(B)$ corresponding to π_B is 0 or 1 (by obs 2 again).

Suppose that $Disj(\nu(B))\restriction_\pi \equiv 1$ (the other case is symmetric), then by construction $\nu(\neg B \vee C)$ represents $Disj(\nu(\neg B)) \vee Disj(\nu(C))$. Since $Disj(\nu(B))\restriction_\pi \equiv 1$ then $Disj(\nu(\neg B))\restriction_\pi \equiv 0$, this is because by obs 2, π_B leads to a leaf in $\nu(B)$ with label 1 and hence π_B leads to a leaf of $\nu(\neg B)$ with label 0.

So $Disj(\nu(\neg B)) \vee Disj(\nu(C))\restriction_\pi \equiv 0$ and since $\pi \supseteq \pi_{\neg B \vee C}$ then there is a leaf in $\nu(\neg B \vee C)$ with label 0. Contradicting the ind. hyp. □

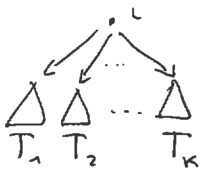
Notice that $\frac{A \vee (B \vee C)}{(A \vee B) \vee C}$ has 7 different subformulas $\{A, B, C, B \vee C, A \vee B, A \vee (B \vee C), (A \vee B) \vee C\}$ so just to be safe we set $k \leq \frac{n}{7}$ so we are able to find π with the properties above for all formulas we need in this rule.

- K-evaluations are well-behaved under restrictions -

Given a tree $T \in CMDT_n$ and $\alpha \in \mathcal{M}_n$, we define $T\restriction_\alpha \in CMDT_n$

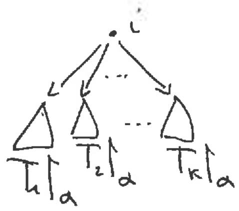
(i) if $|T| = 1$, i.e. the tree has just one node, then $T\restriction_\alpha = T$;

(ii) if $T =$



; • if i is not touched by α then

$T\restriction_\alpha =$



• if $(i,j) \in \alpha$ for some j, then some of the edges going out from i has label (i,j) and let T_j its sub-tree. then $T\restriction_\alpha = T_j\restriction_\alpha$.

observation 3: let $\nu : \Gamma \longrightarrow CMDT_n$ a K-evaluation of a set of formulas Γ and let $\alpha \in \mathcal{M}_n$. Then $\nu' : \Gamma\restriction_\alpha \longrightarrow CMDT(V\restriction_\alpha)$ defined as follows is a K-evaluation.

$\nu'(0)$ is the tree with a single node with label 0, same for $\nu'(1)$ and label 1

For a non-trivial $F\restriction_\alpha \in \Gamma\restriction_\alpha$ with $F \in \Gamma$ we set $\nu'(F\restriction_\alpha) = \nu(F)\restriction_\alpha$.

proof sketch: $\nu'(x_{ij})$ has the correct form (check).

given a tree $T \in CMDT_n$ let T^c be the same tree as T but with labels of the leaves exanged from 0 to 1 and viceversa.

$$\nu'(\neg F\restriction_\alpha) = \nu(\neg F)\restriction_\alpha = \nu(F)^c\restriction_\alpha = (\nu(F)\restriction_\alpha)^c = \nu'(F\restriction_\alpha)^c.$$

(exercise)

If $F \in \Gamma$ is a disj with merged form $\bigvee_{i \in I} F_i$, by hyp. $\nu(F)$ represents $\bigvee_{i\in I} Disj(\nu(F_i))$, then (exercise) $\nu(F)\restriction_\alpha$ represents $\bigvee_{j \in I} Disj(\nu(F_i))\restriction_\alpha$

$$\equiv \bigvee_{i \in I} Disj(\nu(F_i)\restriction_\alpha).$$

(exercise)

So $\nu'(F\restriction_\alpha)$ represents $\bigvee_{i\in I} Disj(\nu'(F\restriction_\alpha))$.

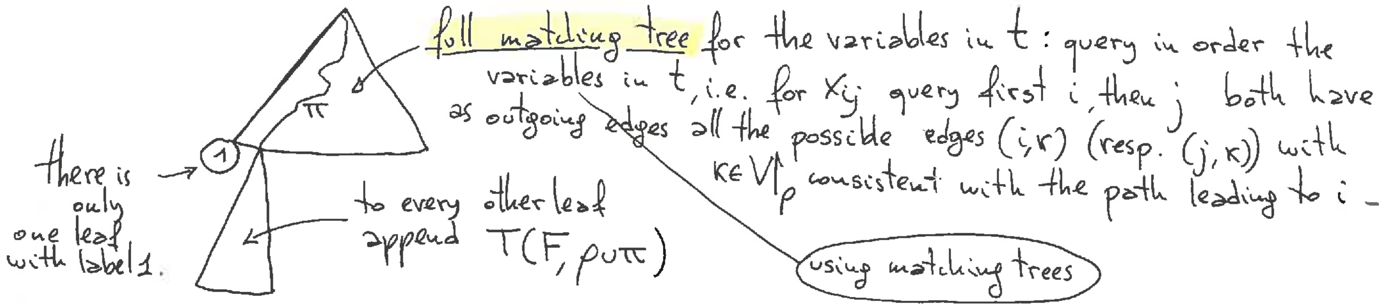
□

- Canonical Matching Decision Trees -

Given a matching disjunction $F = t_1 \vee \dots \vee t_m$, and a matching $\rho \in \mathcal{M}_n$, the **canonical matching decision tree** $T(F,\rho)$ is the following tree in $CMDT(V\restriction_\rho)$ representing $F\restriction_\rho$: fix an ordering on the terms of F and fix an ordering on the variables of F,

(i) if $F\restriction_\rho \equiv 0$ then $T(F,\rho)$ is a single node labeled 0, analogously for $F\restriction_\rho \equiv 1$

(ii) if $F\restriction_\rho \not\equiv 0$ and $F\restriction_\rho \not\equiv 1$ let t be first non-zero term in $F\restriction_\rho$ then $T(F,\rho)$ is constructed as follows:



example 1: the trees we used in the definition of k-evaluations for x_{ij} are canonical.

example 2: $P = \{1,2,3,4,5\}$ $H = \{6,7,8,9\}$

$$F = (x_{17} \wedge x_{38}) \vee (x_{16} \wedge x_{27}) \vee (x_{56} \wedge x_{49}) \vee (x_{16} \wedge x_{59})$$

$\rho = \{(1,6)\}$ suppose that the terms and vars are ordered according the way they are written in F above.

Write $T(F,\rho)$.

$F\restriction_\rho = x_{27} \vee x_{49} \vee x_{59}$ $\qquad V\restriction_\rho = \{2,3,4,5,7,8,9\}$