Solutions, exam 2011-01-13

1 a) $\boxed{\text{TRUE}}$

If $A\underline{x} = \underline{y}$ has no solution, means A is singular (= does not have full rank). Hence, $A\underline{x} = \underline{z}$ will have no solution or an infinite number of solutions, depending on if $\underline{z}$ is in the range of A. It can never have a unique solution.

b) $\boxed{\text{TRUE}}$

c) $\boxed{\text{FALSE}}$

A matrix may have 0 as an eigenvalue and still have a full set of linearly independent eigenvectors, and hence be diagonalizable.

d) $\boxed{\text{TRUE}}$

n columnvectors of length m. If $n > m$, cannot be linearly independent.

e) $\boxed{\text{TRUE}}$

If the cols of A are L.I., then A^TA is symmetric positive definite, i.e. all eigenvalues are positive. Singular values: $\sqrt{\ }$ of eigs of A^TA.

2. a)

$$A\underline{x} = 0 \Rightarrow A^TA\underline{x} = 0$$

$$A^TA\underline{x} = 0 \Rightarrow \underline{x}^TA^TA\underline{x} = 0 \Leftrightarrow (A\underline{x})^TA\underline{x} = 0 \Rightarrow A\underline{x} = 0.$$

$$\therefore \quad A\underline{x} = 0 \Leftrightarrow A^TA\underline{x} = 0.$$

For an $m \times n$ matrix A: $\text{rank}(A) + \text{nullity}(A) = n$

$\text{nullity}(A)$: dimension of null space of A.

A^TA is $n \times n$ and $\text{rank}(A^TA) + \text{nullity}(A^TA) = n$

$\Rightarrow \text{rank}(A) = \text{rank}(A^TA)$.

b) $A_{m\times n} = U_{m\times n}\Sigma_{n\times n}V^T_{n\times n}$, $\quad U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$, $V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$u_1, \ldots, u_n$: and $v_1, \ldots, v_n$: orthonormal vectors.

Σ: diagonal matrix with singular values on diagonal

The SVD always exists. The singular values $\sigma_i = \sqrt{\lambda_i}$ where λ_i, $i = 1, \ldots, n$, are eigenvalues of A^TA.

The number of positive singular values gives the rank of A.

3d). Double eigenvalue $\lambda = -1$

Only one eigenvector, $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\lim_{t\to\infty} \frac{u_2(t)}{u_1(t)} = \frac{c_1 + c_2 t}{1/2\, c_2} = \infty \quad \text{if } c_2 \neq 0, \text{ or if } c_2 = 0 \text{ and } c_1 \neq 0.$$

A stable improper node.

4. a) u is prey, v is predator.

b) Critical pts: $(u, v) = (0, 0)$ and $(u, v) = (d/c, a/b)$

$$J(u,v) = \begin{pmatrix} a - bv & -bu \\ cv & cu - d \end{pmatrix}$$

At the strictly positive point:

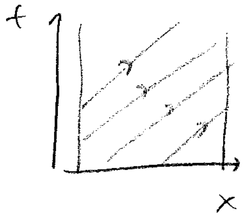
$$J(d/c, a/b) = \begin{pmatrix} 0 & -bd/c \\ ca/b & 0 \end{pmatrix}.$$

Eigs: $\lambda^2 + \frac{ca}{b} \cdot \frac{bd}{c} = 0$

$\lambda = \pm i \sqrt{ad}$

For a linear system, this would be a neutrally stable center. This analysis does not determine the stability of the non linear system in this case. It can either be a spiral or a center.

5 d) contd.



$a > 0$, const

Stencil: 

The stencil:  $\left(u_j^{n+1} = u_j^n - \Delta t\, a \dfrac{u_j^n - u_{j-1}^n}{\Delta x} \right)$

is stable

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(Flip sign of a and this is reversed).

Unstable when stencil not compatible with propagation of information.

Alt: Can use this stencil () but introduce diffusivity by averaging $u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \Delta t\, a \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$.

(Lax-Friedrich).

e) Lax-Equivalence theorem:

"The method is convergent if it is consistent and stable."

The theorem is useful because it is much easier to show that a method is consistent and stable than to directly show that it is convergent

6. $$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

Mult by $P_m(x)$ and integrate:

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \frac{2}{2n+1} \delta_{nm} =$$

$$= a_m \frac{2}{2m+1}$$

$$\Rightarrow \quad a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx.$$