Sols for, 2011-12-20

Problem 2

Uppgift 2 (~~se baksidan~~)

Solution

1. (a)

$$L = \frac{1}{2}y_1^2 + \frac{1}{6}y_2^2 + \lambda\left(y_1 + y_2 - 3\right)$$

(b)

$$\begin{aligned} y_1 + \lambda &= 0 \\ \frac{1}{3}y_2^2 + \lambda &= 0 \\ y_1 + y_2 - 3 &= 0 \end{aligned}$$

2. (a) $L = y^{\mathrm{T}}Ky + \lambda\left(y^{\mathrm{T}}y - 1\right)$

(b)

$$\begin{aligned} \frac{\partial L}{\partial y_1} &= 2 * (k_{11}y_1 + k_{12}y_2 + \lambda y_1) \\ \frac{\partial L}{\partial y_2} &= 2 * (k_{12}y_1 + k_{22}y_2 + \lambda y_2) \\ \frac{\partial L}{\partial \lambda} &= y_1^2 + y_2^2 - 1 \end{aligned}$$

Resulting nonlinear system:

$$\begin{gathered} 2 * (k_{11}y_1 + k_{12}y_2 + \lambda y_1) = 0 \\ 2 * (k_{12}y_1 + k_{22}y_2 + \lambda y_2) = 0 \\ y_1^2 + y_2^2 - 1 = 0 \end{gathered}$$

(c)

$$Ky + \lambda y = 0 \quad \Leftrightarrow \quad Ky = -\lambda y$$

This is an eigenvalue problem!

(d) Bonus/extra questions:

- $y^{\mathrm{T}}y = 1$ means that the eigenvector is normalized.
- $Q = y^{\mathrm{T}}Ky = y^{\mathrm{T}}(-\lambda y) = -\lambda$

(Problem 1 last page).

3. system is

(a) $$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{a}\sin x - \frac{c}{m} y \end{cases}$$

Critical points we get by solving $(\dot{x}, \dot{y}) = 0$

$$\Rightarrow \quad \bar{x}^* = (0,0)$$

(b) Eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - \frac{4g}{a}}\right)$$

hence stable spiral if $\frac{c}{m} > 0$, $\frac{g}{a} > \left(\frac{c}{2m}\right)^2$

(c) Stable node if $\frac{g}{a} < \left(\frac{c}{2m}\right)^2$

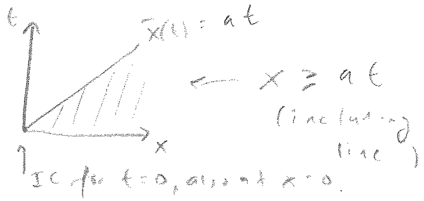
(d) The limit can be obtained as the slopes of the eigenvectors

$$\begin{pmatrix} -\lambda & 1 \\ -q & -p-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

$$\Rightarrow \lim_{t\to\infty} \frac{u_2}{u_1} = \lambda_{1,2}$$

4. $u_t + a u_x = -cu$, $u(x,0) = g(x)$, $u(0,t) = f(t)$, $0 < x < \infty$, $t \geq 0$.

a) $\frac{dx}{dt} = a$, $\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\underbrace{\frac{dx}{dt}}_{a} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -cu$



Determined by initial condition for $x \geq at$, by BC for $x < at$.

b) $u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} a(u_{j+1}^n - u_{j-1}^n) - c u_j^n$

Plug in $u_j^n = \hat{u}_k^n e^{ik2\pi x_j} = \hat{u}_k^n e^{i\bar{k}x_j}$, $\bar{k} = 2\pi k$

$\Rightarrow \hat{u}_k^{n+1} e^{i\bar{k}x_j} = [1 - \frac{\Delta t}{2\Delta x} a \underbrace{(e^{i\bar{k}\Delta x} - e^{-i\bar{k}\Delta x})}_{2i\sin(\bar{k}\Delta x)} - \Delta t \cdot c]\, \hat{u}_k^n e^{i\bar{k}x_j}$

$$\hat{u}_k^{n+1} = \underbrace{\left(1 - \Delta t\, c - i \underbrace{\frac{\Delta t}{\Delta x}}_{\mu} \cdot a \cdot \sin(\bar{k}\Delta x)\right)}_{\hat{Q}(k)} \hat{u}_k^n$$

$|\hat{Q}(k)|^2 = (1 - \Delta t\, c)^2 + \mu^2 a^2 \sin^2(k\Delta x)$

i) $c = 0 \Rightarrow |\hat{Q}(k)|^2 = 1 + \mu^2 a^2 \sin^2(k\Delta x)$ (For stability, need $|\hat{Q}(k)| \leq 1 \ \forall k$)

$\therefore$ The method is unstable $\forall\, \Delta t > 0$.

ii) $c > 0$. $|\hat{Q}(k)|^2 \leq (1 - \Delta t\, c)^2 + \frac{\Delta t^2}{\Delta x^2} \cdot a^2 \leq 1$

$\Rightarrow (c^2 + \frac{a^2}{\Delta x^2})\Delta t^2 - 2c\Delta t + 1 \leq 1$

$\Delta t\left((c^2 + \frac{a^2}{\Delta x^2})\Delta t - 2c\right) \leq 0$

$\therefore \underline{\Delta t \leq \frac{2c}{c^2 + a^2/\Delta x^2}}$ for stability

5 a) $\frac{\partial \eta}{\partial t} = \sqrt{\frac{g}{d}} \frac{\partial}{\partial x}\left(d\eta + \frac{3}{4}\eta^2 + \frac{1}{2}\sigma \frac{\partial^2 \eta}{\partial x^2}\right)$, $\eta = Lu$, $x = Lx'$, $t = Tt'$:

Also write $d = \bar{d} \cdot L$, and $\sigma = \bar{\sigma} L^3$.

$\frac{L}{T}\frac{\partial u}{\partial t'} = \sqrt{\frac{g}{\bar{d}L}} \cdot \frac{1}{L} \frac{\partial}{\partial x'}\left(L^2 \bar{d} u + \frac{3}{4}L^2 u^2 + \frac{1}{2}\bar{\sigma} \cdot L^3 \cdot \frac{L}{L^2}\frac{\partial^2 u}{\partial x'^2}\right)$

$\frac{\partial u}{\partial t'} = T \cdot \sqrt{\frac{g}{\bar{d}L}} \cdot \frac{\partial}{\partial x'}\left(\bar{d}u + \frac{3}{4}u^2 + \frac{1}{2}\bar{\sigma}\frac{\partial^2 u}{\partial x'^2}\right)$. Pick $L = d$, and so $\bar{d} = 1$.

Pick $T = \sqrt{d/g}$

$\therefore \frac{\partial u}{\partial t'} = \frac{\partial}{\partial x'}\left(u + \frac{3}{4}u^2 + \frac{1}{2}a\frac{\partial^2 u}{\partial x'^2}\right)$ $\rightarrow a = \bar{\sigma} = \sigma/L^3 = \sigma/d^3$.

b) Introduce the expansion, for non-linear term introduce e.g: $\sum \widehat{\left(u^N \frac{\partial u^N}{\partial x}\right)}_k e^{2\pi i k x/L}$

Insert, use orthogonality. Introduce t-s, e.g. forward Euler.

How to compute coeffs of non-linear term: IFFT of $\hat{u}_k \rightarrow u$, IFFT of $ik\hat{u}_k \rightarrow u_x$

Multiply in real space FFT to Fourier space to get pseudospectral approx of $\widehat{\left(u^N \frac{\partial u^N}{\partial x}\right)}_k$.

6: $b_k = \frac{1}{N}\sum_{j=0}^{N}(f_j + i g_j)e^{-i2\pi jk/N} = \frac{1}{N}\sum_{j=0}^{N} f_j e^{-i2\pi jk/N} + i\cdot\frac{1}{N}\sum_{j=0}^{N} g_j e^{-i2\pi jk/N} =$

$= c_k + i d_k$

$\overline{b_{N-k}} = \overline{\frac{1}{N}\sum_{j=0}^{N}(f_j + i g_j)e^{-i2\pi j(N-k)/N}} = \frac{1}{N}\sum_{j=0}^{N} f_j \overline{e^{i2\pi jk/N}} - \frac{i}{N}\sum g_j \overline{e^{i2\pi jk/N}}$

$[e^{-i2\pi j} = 1]$ f and g real

$= \frac{1}{N}\sum_{j=0}^{N} f_j e^{-i2\pi jk/N} - i\frac{1}{N}\sum_{j=0}^{N} g_j e^{-i2\pi jk/N} = c_k - i d_k.$

$\frac{1}{2}(b_k + \overline{b_{N-k}}) = \frac{1}{2}(c_k + i d_k + c_k - i d_k) = c_k$

$\frac{i}{2}(\overline{b_{N-k}} - b_k) = \frac{i}{2}(c_k - i d_k - (c_k + i d_k)) = -\frac{2i^2}{2}d_k = d_k$

$\therefore \quad c_k = \frac{1}{2}(b_k + \overline{b_{N-k}}), \quad d_k = \frac{i}{2}(\overline{b_{N-k}} - b_k)$

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1. a) False. (can still have an eigenvalue of 0)

   b) True. ($Q_3^T Q_3 = (Q_1 Q_2)^T Q_1 Q_2 = Q_2^T \underbrace{Q_1^T Q_1}_{I} Q_2 = Q_2^T Q_2 = I.$)

   c) True. (All $\sigma_i > 0$, A has full rank.)

   d) True. (All $\lambda_i > 0$, i.e. A^{-1} exist. $A\underline{x} = \lambda x \Rightarrow x = \lambda A^{-1}x \Rightarrow A^{-1}x = \frac{1}{\lambda}\underline{x}$, $1/\lambda > 0$, i.e. P.D.).