Exam MatMod, 2012-02-18.

1. a) True, b) False (only true if A and B commutes), c) False. (only true if cols of A L.I), d) True

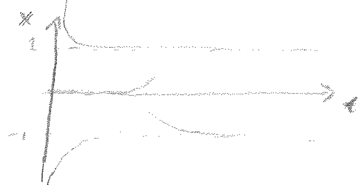
2. a) It always exist. U & V: orthonormal columns, Σ: diagonal, singular values on diagonal. All $\sigma_i > 0$ when all cols of A L.I (i.e A^TA is SPD)

b) $A^TA = (U\Sigma V^T)^T U\Sigma V^T = V \underbrace{\Sigma^T}_{=\Sigma} \underbrace{U^TU}_{I_n} \Sigma V^T = V \Sigma^2 V^T$; i.e V diagonalizes A^TA; eigs are diag vals of Σ^2.

c) A has L.I cols $(A^TA)^{-1}A^T = V\Sigma^{-1}U^T$

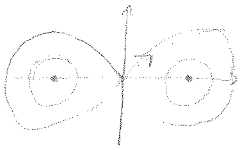
d) $A^TA\underline{x} = A^T\underline{b} \Rightarrow V\Sigma^2V^T\underline{x} = V\Sigma U^T\underline{b} \Rightarrow \underline{x} = V\Sigma^{-1}U^T\underline{b}$.

3. a) $\frac{dx}{dt} = x(1-x^2) \to$ critical pts $x=0$, $x=\pm 1$. $f(x) = x - x^3$, $df/dx = 1-3x^2$; $=1$ at $x=0$, -2 at $x=\pm 1$. $x=0$ unstable. $x=\pm 1$ stable.



b) crit pts: $(0,0)$, $(-1,0)$, $(1,0)$. $J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda = \pm 1$ unstable

$J(-1,0) = J(1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $\lambda = \pm i\sqrt{2}$: neutrally stable (centers).



4. a) (1) $\Rightarrow (i\omega + a i\beta)\hat{u}e^{i(\omega t + \beta x)} = 0 \Rightarrow \omega = -a\beta$. , $u(x,t) = \hat{u}\cdot e^{i\beta(-at+x)}$

Prop. w speed a, no decay of sol. (2) $\omega = \epsilon\beta^2 i$, $u(x,t) = \hat{u}\cdot e^{-\epsilon\beta^2 t} e^{i\beta x}$, i.e decays in time and wave does not move

b) $\bar{u}_t + a\bar{u}_x + \frac{\Delta t}{2}\bar{u}_{tt} - a\frac{\Delta x}{2}\bar{u}_{xx} = O(\Delta t^2, \Delta x^2)$. Using $u_{tt} = a^2u_{xx} \Rightarrow u_t + au_x + \frac{a}{2}(a\Delta t - \Delta x)u_{xx} = O(\Delta t^2, \Delta x^2)$

2nd order approx for $u_t + au_x = \varepsilon u_{xx}$ with $\varepsilon = \frac{a}{2}(\Delta x - a\Delta t)$.

Diffusive errors. $\varepsilon = 0$ for $\Delta t = \Delta x/a$

5. a) $T = 1/k$, $L = \sqrt{D/k}$ b) Introduce expansion of nonlinear term: $u^2(x,t) = \sum_{k=-N/2}^{N/2-1} \widehat{(uu)}_k e^{i2\pi kx}$

Insert, mult by $e^{-i2\pi ex}$, integrate, use orthogonality

i) $\Rightarrow \frac{d\hat{u}_k}{dt} = (-(2\pi)^2k^2 + 1)\hat{u}_k - \widehat{(uu)}_k$, $k = -N/2, \dots, N/2-1$.

ii) Forward Euler $t_n = n\Delta t$, $\hat{u}_k^n = \hat{u}_k(t_n) \Rightarrow \hat{u}_k^{n+1} = \hat{u}_k^n + \Delta t\,(-4\pi^2k^2 + 1)\hat{u}_k^n + \widehat{(uu)}_k^n$ (*).

a) Given initial condition $u(x,0)$, do FFT to compute $\hat{u}_k^0$, $k = -N/2, \dots, N/2-1$

For $n = 0, 1, 2, \dots$: Multiply $u \cdot u$ in real space, do FFT to find $\widehat{(uu)}_k^n$.

Evaluate $\hat{u}_k^{n+1}$ using (*) $\forall k$

Do an IFFT to find u at time level $n+1$.

6). $f(x) = \sum_{n=0}^{\infty} a_n T_n(x). \Rightarrow \int_{-1}^{1} f(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = a_m \cdot c_m$,

where $c_0 = \pi$ and $c_m = \frac{\pi}{2}$ if $m \neq 0$,

$\therefore a_m = \frac{1}{c_m} \int_{-1}^{1} f(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx$.