Homework 5. Dynamic programming

(The exercise was formulated by Anders Szepessy using ideas from Jonathan Goodman, NYU.)
The following is a qualitative model, unrealistic in several ways in order to be simple. A loan of 300 000 kr is to be repaid in 30 years. Each year, we add some interest and make a payment. If our interest rate at year t is $r(t)$, and the outstanding balance is $B(t)$, then, at year t we first add interest, replacing $B(t)$ by $B'(t) = (1+r(t))B(t)$. Next we make a prorated payment, $P(t) = 3B'(t)/(33-t)$. This leaves $B(t+1) = B'(t) - P(t)$. The goal is to minimize the total payment in two cases.

5a. *Deterministic control.* Assume that the floating interest rate $R(t)$ is the given 10-year periodic function which is linear between the points $R(0) = 0.05$, $R(2.5) = 0.1$, $R(7.5) = 0.01$, $R(10) = 0.05$, ... Our interest rate r is determined as follows. At year 0, $r(0) = R(0) = 0.05$. However, $r(t)$ remains fixed until we "refinance". Whenever we refinance, $r(t)$ is replaced by $R(t)$. We are allowed to refinance up to four times during this 30 year period. We want to do so to minimize our total payments, $P(0) + P(1) + \cdots + P(30)$. Set this up as a dynamic programming problem. Identify the state space. Write a computer program to solve it by dynamic programming. Use interpolation to estimate unknown values of the cost to go function from previously computed values. Identify the optimal control.

5b. *Stochastic control.* We assume now that the deterministic floating interest rate $R(t)$ is replaced by a stochastic interest that fluctuates between 1% and 10% (annualized). Now it is 5%. Each year it may move up or down by 2%, except that it is blocked from moving outside its range. It moves in either direction with probability 1/3 and also remains unchanged with this probability. All movements are independent. This interest rate is $R(t)$. Otherwise, everything is as in exercise 5a, and we want to minimize the expected total payment $E[P(0) + P(1) + \cdots + P(30)]$. Formulate this as a dynamic programming problem and extend the computer program in exercise 5a to solve it. What is the optimal control?