On the Approximation Resistance of a Random Predicate

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Johan Håstad Approximation resistance of CSPs



2 Approximation resistance

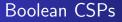




History of talk

Originally given at the APPROX-conference in Princeton, August 2007.

Survey of the general area, stating results but giving very few details. Ask for details.



Constraint Satisfaction Problems where each input is a bit. Same predicate appears P in all constraints. A *k*-ary predicate P that accepts *t* of the 2^{*k*} inputs.

Most famous example of CSPs, k-sat

Disjunctions of k literals, m constraints, n variables.

2-Sat:
$$(x_1 \lor \overline{x}_2) \land (x_2 \lor x_7) \land \ldots \land (\overline{x}_1 \lor \overline{x}_{11}).$$

 $k = 2, t = 3$

3-Sat: $(x_1 \lor \overline{x}_2 \lor x_3) \land (x_2 \lor x_7 \lor \overline{x}_8) \land \ldots \land (\overline{x}_1 \lor x_8 \lor \overline{x}_{11}).$ k = 3, t = 7.



System of linear equations modulo 2 with at most three variables in each equation.

$$\begin{cases} x_1 + x_2 + x_3 &= 1\\ x_1 + x_2 &= 1\\ x_1 + x_2 + & x_4 = 1\\ & x_2 + & x_4 = 0\\ x_1 + & x_3 + x_4 = 0\\ & & x_2 + x_3 + x_4 = 1\\ x_1 + & & x_3 &= 0 \end{cases} \mod 2$$

m equations *n* variables.

Many predicates, majority, not-all-equal etc

We have 2^{2^k} predicates on k inputs. It is specified by an answer on each k bit string.

Equivalent predicates

Please note that negations are allowed for free, and so are permutations of the inputs.

We get families of equivalent predicates, each containing up to $2^k \cdot k!$ different predicates.

The number of families is $2^{2^k(1-o(1))}$.

Number of different predicates

Can be calculated by computer.

k	2	3	4	
Predicates	14	254	65534	Only counting non-constant
Non-EQ	3	16	400	

predicates.



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Max-CSP: Find the assignment that satisfies that maximum number of constraints.



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Max-CSP: Find the assignment that satisfies that maximum number of constraints.

Given a list of m k-tuples of literals find an assignment that makes as many as possible of the resulting k-tuples of bits satisfy P.

Approximation ratio

An algorithm has approximation ratio α if for any instance

 $\frac{\text{Value of found solution}}{\text{Value of optimal solution}} \geq \alpha$

For randomized algorithms, expectation over internal coinflips, always worst case inputs.

Easy result for Max-3Sat

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_7 \vee \bar{x}_8) \wedge \ldots \wedge (\bar{x}_1 \vee x_8 \vee \bar{x}_{11})$$

A random assignment satisfies each clause with probability 7/8.

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How do we do this deterministically?

Easy result for Max-3Lin

$$\begin{cases} x_1 + x_2 + x_3 &= 1\\ x_1 + x_2 &= 1\\ x_1 + x_2 + & x_4 = 1\\ & x_2 + & x_4 = 0\\ x_1 + & x_3 + x_4 = 0\\ & & x_2 + x_3 + x_4 = 1\\ x_1 + & & x_3 &= 0 \end{cases} \mod 2$$

A random assignment satisfies each clause with probability 1/2. We get an 1/2-approximation algorithm.



- It is easy to approximate Max-P within $t2^{-k}$.
- A random assignment satisfies on the average $t2^{-k}m$ constraints and this can be found deterministically.



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- A random assignment satisfies on the average $t2^{-k}m$ constraints and this can be found deterministically.
- The trivial approximation ratio.

Approximation resistance

A predicate *P* is approximation resistant if $\forall \epsilon > 0$ it is hard to approximate max-CSP(*P*) within $\epsilon + t2^{-k}$.

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A predicate P is approximation resistant if $\forall \epsilon > 0$ it is hard to approximate max-CSP(P) within $\epsilon + t2^{-k}$.

A predicate *P* is approximation resistant on satisfiable instances if $\forall \epsilon > 0$ it is hard distinguish instances where we can satisfy all constraints from those where we can only satisfy a fraction $\epsilon + t2^{-k}$ of the constraints.

Hereditary properties

A predicates P is hereditary approximation resistant if whenever $P(x) \Rightarrow Q(x)$ then Q is also approximation resistant.

My view

Approximation resistance on satisfiable instances is possibly the ultimate hardness for a CSP.

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Are there such predicates?

Binary constraints

Constraints on two variables, k = 2.

Semidefinite programming [GW95,LLZ] gives an .8740 approximation algorithm in general.

There are hence no approximation resistant predicates on two binary variables.

More general fact

Extends to give non-approximation resistance of binary predicates over all domains sizes [H05] and binary constraints.

The case k = 3

Max-3-Lin is hereditary approximation resistant [H01], and this gives all approximation resistant predicates [Z98] on three inputs.

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What does this give for Max-3-Sat, Max-3-Maj, Max-3-NAE?

Satisfiable instances

Max-3-Sat is approximation resistant on satisfiable instances.

Unknown what happens for the "not two ones predicate" on satisfiable instances.

The case k = 4

Partial classification by Hast [H05]. 400 essentially different predicates.

- 79 approximation resistant.
- 275 not approximation resistant.
- 46 not classified.

# Acc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Non-res	1	4	6	19	27	50	50	52	27	26	9	3	1	0	0
Res	0	0	0	0	0	0	0	16	6	22	11	15	4	4	1
Unkn	0	0	0	0	0	0	6	6	23	2	7	1	1	0	0

Satisfiability ignored.

Questions

How common is approximation resistance?

Can we find big classes of approximation resistant predicates?

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Can we find big classes of approximation resistant predicates? What about a random predicate?

Random predicates

A random predicate from space $R_{p,k}$ accepts each input with probability p (and has $t \approx p2^k$).

Is a random predicate for p = 1/2 likely to be approximation resistant?

Predicate P_{ST}^1

A predicate given by a subspace of dimension $l_1 + l_2$ with $k = l_1 + l_2 + l_1 l_2$.

Showed to be approximation resistant by Samorodnitsky and Trevisan [ST00] and hereditary so by Hast [H05].

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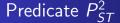
Gives many approximation resistant predicates but is it enough for a random predicate?

A rough calculation

If we accept t inputs the probability that it implies a random predicate is 2^{-t} .

We have at most $k!2^k$ equivalent predications.

Need $t \leq k \log k$ while we have $t \approx 2^{\sqrt{k}}$ for P_{ST}^1 .



A predicate given by a subspace of dimension d with $2^{d-1} < k \le 2^d - 1$.

Assuming the Unique Games Conjecture (UGC) showed to be approximation resistant by Samorodnitsky and Trevisan [ST05].

Unique Games Conjecture

Made by Khot [K02], a binary CSP over a large alphabet L. Constraints are permutations $\pi_{ij}(x_i) = x_j$ for some pairs i, j.

Problem: Distinguish instances where we can satisfy fraction $1 - \epsilon$ from those where we can only satisfy fraction δ .

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A very open conjecture.

Main result

Theorem: Assuming the unique games conjecture a random predicate from $R_{1/2,k}$ is with high probability, for sufficiently large k, approximation resistant.

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Extends to $p = k^{-c}$ for $1/2 \le c \le 1$, $c \approx k2^{-d}$.

Proof 1

Assuming UGC P_{ST}^2 is hereditary approximation resistant. Extending the proof of Samorodnitsky and Trevisan.



Lemma: For $S \subseteq [d]$ functions f_S such that

- One function (almost) unbiased, $|E[f_S(x)]| \leq \delta$.
- No two functions have high common influence, $\max(\inf_i(f_{S_1}), \inf_i(f_{S_2})) \leq \epsilon.$ $\left| E_{x_1...x_d} \left[\prod_{S \subset [d]} f_S(\prod_{i \in S} x_i) \right] \right| \leq \delta + (2^d - 2)\sqrt{\epsilon},$
- i.e. Gowers uniformity norm is small.



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New simpler more direct proof compared to [ST05].

Proof 2

Prove that if Q is random from $R_{1/2,k}$ then it is likely that there is a P_{ST}^2 -equivalent predicate P' such that $P' \Rightarrow Q$.

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Second moment method using only $P_{ST}^2\mbox{-}{\rm equivalent}$ predicates that are very different.



- Approximation resistance is a very strong notion of hardness.
- If the Unique Games Conjecture is true then a vast majority of predicates are approximation resistant.

Open problems

- Prove result without the unique games conjecture.
- Prove approximation resistance on satisfiable instances.
- Olassify more predicates with respect to approximation resistance.