On Length, Width and Space in Resolution

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Outline

1. A Resolution Primer
   - Definitions and Notation
   - Highlights of Research Results
2. Our Contribution: Separation of Space and Width
   - Pebble Games
   - Pebbling Contradictions
   - Sketch of Proof
3. Some Open Problems
   - A List of Some Nice Open Problems
   - A Plausible Line of Attack for the Nicest Problem

Resolution

- Prove tautologies $\Leftrightarrow$ refute unsatisfiable formulas in conjunctive normal form (CNF)
- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Also used in many real-world automated theorem provers

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \ldots \lor a_k$: set of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \land \ldots \land C_m$: set of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)
- Refer to clauses of CNF formula as axioms
  (as opposed to derived clauses)

Some More Notation and Terminology

- Truth value assignment α makes
  - clause true if one literal true
  - CNF formula true if all clauses true
- $F \vDash D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \ldots, n\}$

Resolution Rule

Resolution rule:

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

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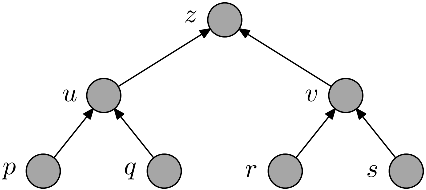
Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

Example CNF Formula

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



- source vertices true
- truth propagates upwards
- but target vertex is false

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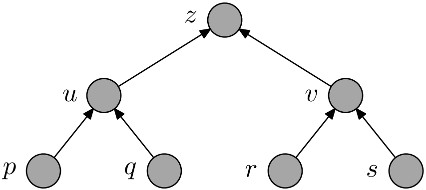

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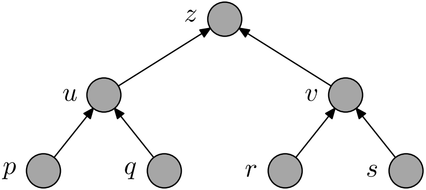

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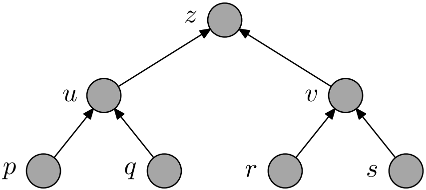

- source vertices true
- truth propagates upwards
- but target vertex is false

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	0
# literals in largest clause	0
# lines on blackboard used	0

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	1
# literals in largest clause	1
# lines on blackboard used	1

p

Write down axiom 1: p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	2
# literals in largest clause	1
# lines on blackboard used	2

p q

Write down axiom 2: q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

p
q
$\overline{p} \lor \overline{q} \lor u$

Write down axiom 5: $\overline{p} \lor \overline{q} \lor u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

p
q
$\overline{p} \lor \overline{q} \lor u$

Infer $\overline{q} \lor u$ from
p and $\overline{p} \lor \overline{q} \lor u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Infer $\overline{q} \vee u$ from p and $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \lor u$

Erase clause $\overline{p} \lor \overline{q} \lor u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$

Infer u from
q and $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$
u

Infer u from
q and $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$
u

Erase clause q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$\overline{q} \lor u$
u

Erase clause q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$\overline{q} \vee u$
u

Erase clause $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

u

Erase clause $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

u
r

Write down axiom 3: r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

u
r
s

Write down axiom 4: s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

u
r
s
$\overline{r} \vee \overline{s} \vee v$

Write down axiom 6: $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

u
r
s
$\overline{r} \lor \overline{s} \lor v$

Infer $\overline{s} \lor v$ from
r and $\overline{r} \lor \overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
r
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Infer $\overline{s} \lor v$ from
r and $\overline{r} \lor \overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
r
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Erase clause r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Erase clause r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Erase clause $\overline{r} \lor \overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$

Infer v from
s and $\overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \lor v$
v

Infer v from
s and $\overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \lor v$
v

Erase clause s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
$\overline{s} \lor v$
v

Erase clause s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
$\overline{s} \lor v$
v

Erase clause $\overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
v

Erase clause $\overline{s} \lor v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$

Write down axiom 7: $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$

Infer $\overline{v} \vee z$ from
u and $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Infer $\overline{v} \vee z$ from
u and $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Erase clause u

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{u} \lor \overline{v} \lor z$
$\overline{v} \lor z$

Erase clause u

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{u} \lor \overline{v} \lor z$
$\overline{v} \lor z$

Erase clause $\overline{u} \lor \overline{v} \lor z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{v} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{v} \lor z$

Infer z from
v and $\overline{v} \lor z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{v} \lor z$
z

Infer z from
v and $\overline{v} \lor z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{v} \vee z$
z

Erase clause v

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor z$

z

Erase clause v

Example Resolution Refutation

1. p
2. q
3. r
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5. $\overline{p} \lor \overline{q} \lor u$
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8. $\overline{z}$

Blackboard bookkeeping	
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# literals in largest clause	3
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$\overline{v} \lor z$

z

Erase clause $\overline{v} \lor z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
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8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

z

Erase clause $\overline{v} \lor z$

Example Resolution Refutation

1. p
2. q
3. r
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5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

z
$\overline{z}$

Write down axiom 8: $\overline{z}$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

z
$\overline{z}$

Infer 0 from
z and $\overline{z}$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	15
# literals in largest clause	3
# lines on blackboard used	5

z
$\overline{z}$
0

Infer 0 from
z and $\overline{z}$

More Formally Speaking. . .

Resolution derivation

Sequence of clause configurations $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

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Resolution refutation of F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that empty clause $0 \in \mathbb{C}_\tau$

Also sometimes referred to as resolution proof of F

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Also sometimes referred to as resolution proof of F

Length, Width and Space

- Length $L(\pi)$ of refutation $\pi : F \vdash 0$
  # distinct clauses in all of π
  (in our example 15)
- Width $W(\pi)$ of refutation $\pi : F \vdash 0$
  # literals in largest clause in π
  (in our example 3)
- Space $Sp(\pi)$ of refutation $\pi : F \vdash 0$
  # clauses in largest clause configuration $\mathbb{C}_t \in \pi$
  (in our example 5)

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{ L(\pi) \}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{ W(\pi) \}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{ Sp(\pi) \}$$

Why Should We Care About These Measures?

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm
- Width: Intimately connected to length and space ☺

Can also give ideas for proof search heuristics

When comparing measures, for simplicity consider mostly *k*-CNF formulas (during this talk)

Results for Length

Easy upper bound: $L(F \vdash 0) \leq 2^{(\#\text{ variables in } F + 1)}$

Theorem (Haken 1985)

Polynomial-size CNF formula family with exponential lower bound on resolution refutation length (pigeonhole principle)

Since then many exponential lower bounds for different formula families

But resolution used widely in practice anyway
Amenable to proof search because of its simplicity

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Connection between Length and Width (1/2)

Trivial upper bound: $W(F \vdash 0) \leq \#$ variables in F

Also, a narrow resolution refutation is necessarily short

For a refutation in width w, bound on length $\leq (2 \cdot \#\text{ variables})^w$
(max # distinct clauses)

Connection between Length and Width (2/2)

There is a kind of converse to this:

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a k-CNF formula F over n variables is

$$W(F \vdash 0) = \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

Proof search heuristic: search for narrow refutations!

Two comments:

- Short and narrow refutation need not be the same one!?
- Bound on width in terms of length essentially optimal (Bonet & Galesi 1999)

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Results for Space

- Space introduced by Esteban & Torán (1999)
- Maximal # clauses in memory while verifying proof—related to performance of proof search algorithms
- Easy upper bound: $Sp(F \vdash 0) \leq$ size of F, or more precisely $\leq \min(\#\text{ variables in } F, \#\text{ clauses in } F) + \mathcal{O}(1)$
- Many lower bounds proven, e.g. polynomial-size k-CNF formula families matching upper bounds above up to multiplicative constant (Alekhnovich et al. 2000, Torán 1999)
- Also, all space lower bounds turned out to match width lower bounds! True in general?

Connection between Space and Width

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?
Are they in fact the same measure asymptotically?

Or can they be separated?
I.e., is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

Separation of Space and Width

Theorem (Nordström 2006)

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- *refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and*
- *refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.*

Second part of talk: overview of this result

Try to convey main ideas—will gloss over all gory details

Another Space Measure: Variable Space

Clause space $Sp(\cdot)$

clauses on blackboard $|\mathbb{C}|$

Variable space $VarSp(\cdot)$

Total # literals on blackboard $\sum_{C \in \{\mathbb{C}\}} |C|$

Which space measure is "the right one"?

Potentially long discussion...

Short answer: both are interesting

Trade-off Results Involving Variable Space (1/2)

Theorem (Ben-Sasson 2002)

Exists family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ such that

- $L(F_n \vdash 0) = \Theta(n)$,
- $W(F_n \vdash 0) = \mathcal{O}(1)$,
- $Sp(F_n \vdash 0) = \mathcal{O}(1)$, *but*
- $VarSp(F_n \vdash 0) = \Theta(n/\log n)$.

Corollary (Ben-Sasson 2002)

For any refutation π of F_n, $Sp(\pi) \cdot W(\pi) = \Omega(n/\log n)$.

Proof of corollary.

For any refutation π, $Sp(\pi) \cdot W(\pi) \geq VarSp(\pi)$. □

Trade-off Results Involving Variable Space (1/2)

Theorem (Ben-Sasson 2002)

Exists family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ *of size* $\mathcal{O}(n)$ *such that*

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For any refutation π *of* F_n, $Sp(\pi) \cdot W(\pi) = \Omega(n/\log n)$.

Proof of corollary.

For any refutation π, $Sp(\pi) \cdot W(\pi) \geq VarSp(\pi)$. □

Trade-off Results Involving Variable Space (2/2)

(Incorrectly Stated) Theorem (Hertel & Pitassi 2007)

Exists family of CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ such that

- $VarSp(F_n \vdash 0) = \Theta\left(\sqrt[3]{n}\right)$,
- $VarSp(\pi) = VarSp(F_n \vdash 0) \Rightarrow L(\pi) = \exp\left(\Omega\left(\sqrt[3]{n}\right)\right)$,
- *adding just 3 more bits of memory can get* π' *with* $VarSp(\pi') = VarSp(F_n \vdash 0) + 3 = \mathcal{O}\left(\sqrt[3]{n}\right)$ *and* $L(\pi') = \mathcal{O}(n)$.

Technical issues to be resolved:

- stated for refutations but proved for derivations (probably true in both cases but some work needed)
- need added condition that F_n is minimally unsatisfiable for theorem to be interesting (probably is the case)

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Pebbles Games

One-player game played on directed acyclic graphs (DAGs)

- Devised for studying programming languages and compiler construction
- Have found a variety of applications in complexity theory

Conventions

- $V(G)$ denote the vertices of a DAG G
- vertices with indegree 0 are sources
- vertices with outdegree 0 are targets

Only consider DAGs with single target z and indegree 2 for all non-source vertices

Definition of Black-White Pebble Game

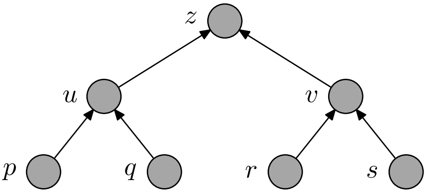
Start with all vertices of DAG G empty

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

Goal: get black pebble on target vertex of G with no other pebbles in G, using as few pebbles as possible

Studied by Cook & Sethi (1976) and many others

Example Pebbling and Pebbling Price



- Cost of pebbling:
  max # pebbles simultaneously in *G*
  (in our example 4)
- Black-white pebbling price *BW-Peb*(*G*) of DAG *G*:
  minimal cost of any pebbling
- (Black) pebbling price *Peb*(*G*):
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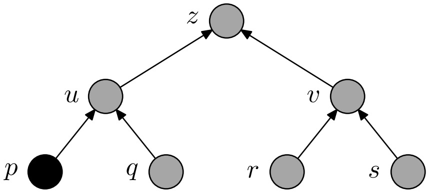

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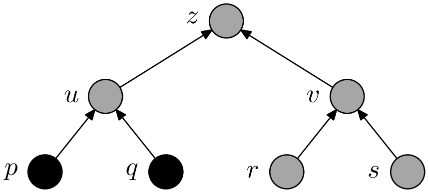

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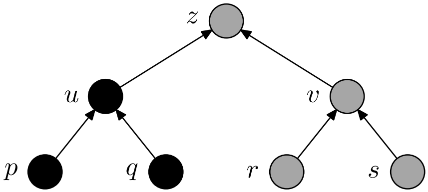

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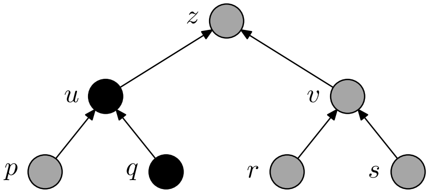

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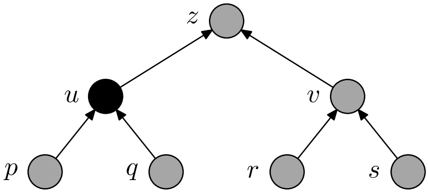

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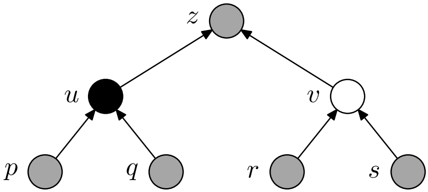

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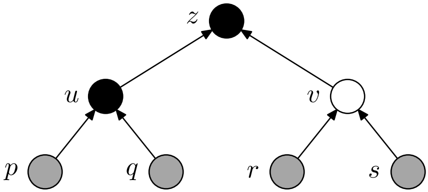

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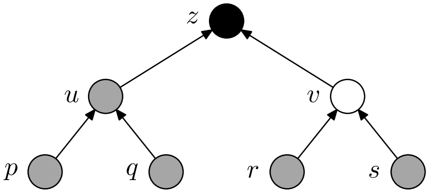

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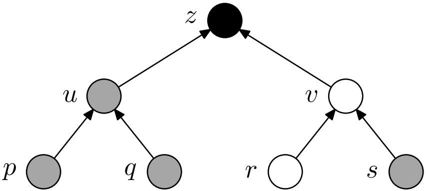

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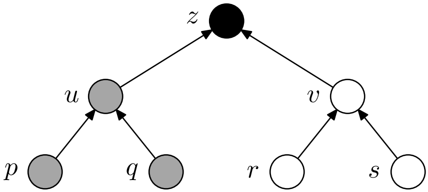

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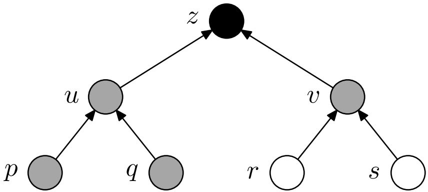
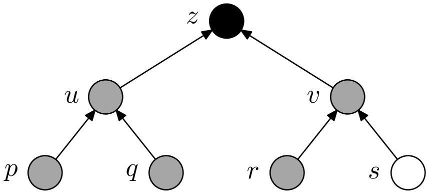


- Cost of pebbling:
  max # pebbles simultaneously in *G*
  (in our example 4)
- Black-white pebbling price *BW-Peb*(*G*) of DAG *G*:
  minimal cost of any pebbling
- (Black) pebbling price *Peb*(*G*):
  minimal cost of pebbling using black pebbles only

Example Pebbling and Pebbling Price



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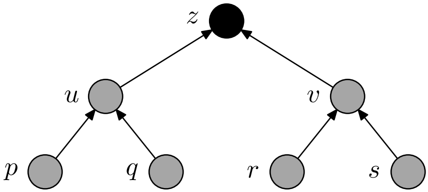

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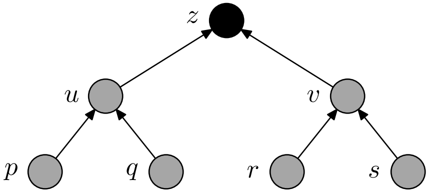
Example Pebbling and Pebbling Price



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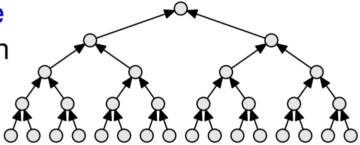
Pebbling Price of Binary Trees

Let T_h denote complete binary tree of height h considered as DAG with edges directed towards root



- Pebbling price of T_h is

$$Peb(T_h) = h + 2$$

(easy induction over the tree height)

- Black-white pebbling price is

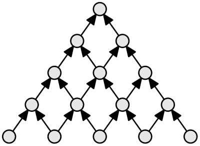
$$BW\text{-}Peb(T_h) = \left\lfloor \frac{h}{2} \right\rfloor + 3 = \Omega(h)$$

(Lengauer & Tarjan 1980)

Pebbling Price of Pyramids

Let Π_h denote pyramid graph of height h considered as DAG with edges directed towards root

- $\mathit{Peb}(\Pi_h) = h + 2$
  (Cook 1974)
- $\mathit{BW\text{-}Peb}(\Pi_h) = \left\lfloor \frac{h}{2} \right\rfloor + \mathcal{O}(1) = \Omega(h)$
  (Klawe 1985)



DAG Size-Pebbling Price Trade-off

- Binary tree of size n has pebbling price $\Theta(\log n)$
- Pyramid of size n has pebbling price $\Theta\left(\sqrt{n}\right)$

Pebbling Contradiction

CNF formula encoding pebble game on DAG G with unique target z and all non-source vertices having indegree 2

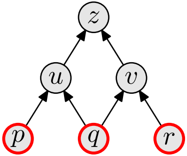
Associate d variables $v_1, \ldots, v_d$ with every vertex $v \in V(G)$

The dth degree pebbling contradiction Peb^d_G over G says that:

- All source vertices have at least one true variable
- Truth propagates upwards according to pebbling rules
- For the target z all variables are false

Studied by Bonet et al. (1998), Raz & McKenzie (1999), Ben-Sasson & Wigderson (1999) and others

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
&(p_1 \lor p_2) \\
\land\ &(q_1 \lor q_2) \\
\land\ &(r_1 \lor r_2) \\
\land\ &(\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
\land\ &(\overline{p}_1 \lor \overline{q}_2 \lor u_1 \lor u_2) \\
\land\ &(\overline{p}_2 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
\land\ &(\overline{p}_2 \lor \overline{q}_2 \lor u_1 \lor u_2) \\
\land\ &(\overline{q}_1 \lor \overline{r}_1 \lor v_1 \lor v_2) \\
\land\ &(\overline{q}_1 \lor \overline{r}_2 \lor v_1 \lor v_2)
\end{aligned}
$$

$$
\begin{aligned}
\land\ &(\overline{q}_2 \lor \overline{r}_1 \lor v_1 \lor v_2) \\
\land\ &(\overline{q}_2 \lor \overline{r}_2 \lor v_1 \lor v_2) \\
\land\ &(\overline{u}_1 \lor \overline{v}_1 \lor z_1 \lor z_2) \\
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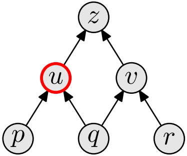
\land\ &(\overline{u}_2 \lor \overline{v}_2 \lor z_1 \lor z_2) \\
\land\ &\overline{z}_1 \\
\land\ &\overline{z}_2
\end{aligned}
$$

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
& (p_1 \lor p_2) \\
\land\ & (q_1 \lor q_2) \\
\land\ & (r_1 \lor r_2) \\
\land\ & (\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
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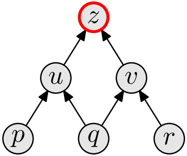
\land\ & (\overline{u}_2 \lor \overline{v}_2 \lor z_1 \lor z_2) \\
\land\ & \overline{z}_1 \\
\land\ & \overline{z}_2
\end{aligned}
$$

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
& (p_1 \lor p_2) \\
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\land\ & (\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
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\land\ & (\overline{u}_2 \lor \overline{v}_2 \lor z_1 \lor z_2) \\
\land\ & \overline{z}_1 \\
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\end{aligned}
$$

Pebbling Contradictions Easy w.r.t. Length and Width

Peb^d_G is an unsatisfiable (2+d)-CNF formula with

- $d \cdot |V(G)|$ variables
- $\mathcal{O}\big(d^2 \cdot |V(G)|\big)$ clauses

Can be refuted by deriving $\bigvee_{i=1}^{d} v_i$ for all $v \in V(G)$ inductively in topological order and resolving with target axioms $\overline{z}_i$, $i \in [d]$

It follows that

- $L(F \vdash 0) = \mathcal{O}\big(d^2 \cdot |V(G)|\big)$
- $W(F \vdash 0) = \mathcal{O}(d)$

(Ben-Sasson et al. 2000)

What about Pebbling Contradictions and Space?

Upper bounds:

- Arbitrary DAGs G
  optimal black pebbling of G + proof from previous slide:
  $Sp\left(Peb_G^d \vdash 0\right) \leq Peb(G) + \mathcal{O}(1)$
- Binary trees T_h
  improvement by Esteban & Torán (2003):
  $Sp\left(Peb_{T_h}^2 \vdash 0\right) \leq \left\lceil \frac{2h+1}{3} \right\rceil + 3 = \frac{2}{3} Peb(T_h) + \mathcal{O}(1)$
- Only one variable / vertex
  Ben-Sasson (2002):
  $Sp\left(Peb_G^1 \vdash 0\right) = \mathcal{O}(1)$ for arbitrary G

No lower bounds on space for $d \geq 2$ previously known

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Rephrasing Our Result

Theorem (Nordström 2006)

Let $Peb^d_{T_h}$ denote the pebbling contradiction of degree $d \geq 2$ defined over the complete binary tree of height h. Then the space of refuting $Peb^d_{T_h}$ in resolution is $Sp(Peb^d_{T_h} \vdash 0) = \Theta(h)$.

Previous theorem follows as corollary, since height grows logarithmically in tree size

Proof Idea

Prove lower bounds on space of $\pi : Peb^d_G \vdash 0$ by

1. Interpreting clause configurations $\mathbb{C}_t$ in terms of black and white pebbles on T_h
2. Showing that if $\mathbb{C}_t$ corresponds to N pebbles it contains at least N clauses (if $d \geq 2$)
3. Establishing that resolution refutations induce black-white pebblings under this interpretation

Then some $\mathbb{C}_t \in \pi$ must induce $BW\text{-}Peb(T_h)$ pebbles

$$\Downarrow$$

$$|\mathbb{C}_t| \geq BW\text{-}Peb(T_h) = \Omega(h)$$

$$\Downarrow$$

$$Sp\left(Peb^d_{T_h} \vdash 0\right) = \Omega(h)$$

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$$\Downarrow$$

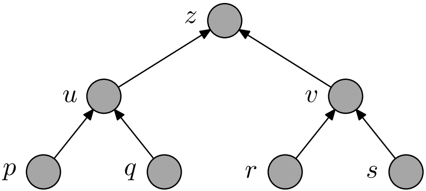
$$|\mathbb{C}_t| \geq \textit{BW-Peb}(T_h) = \Omega(h)$$

$$\Downarrow$$

$$Sp\left(Peb^d_{T_h} \vdash 0\right) = \Omega(h)$$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



Developing an Intuition for Black Pebbles

1. p
2. q
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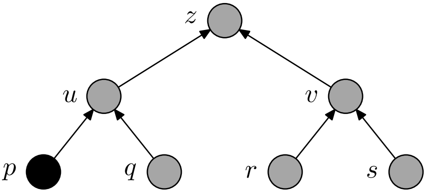
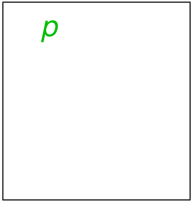
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$





Download axiom 1: p

Developing an Intuition for Black Pebbles

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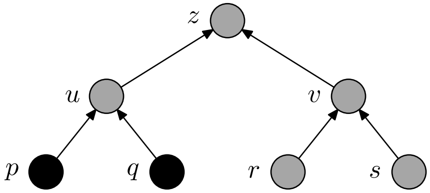
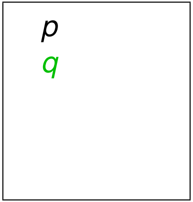
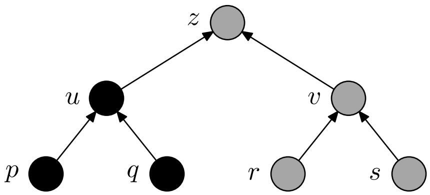




Download axiom 2: q

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



p
q
$\overline{p} \lor \overline{q} \lor u$

Download axiom 5: $\overline{p} \lor \overline{q} \lor u$

Developing an Intuition for Black Pebbles

1. p
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3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
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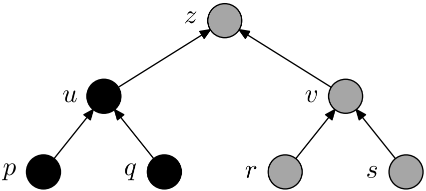

p
q
$\overline{p} \vee \overline{q} \vee u$

Infer $\overline{q} \vee u$ from
p and $\overline{p} \vee \overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
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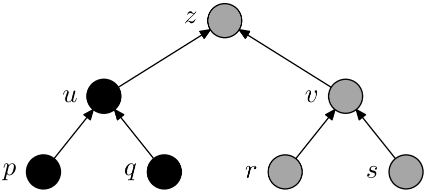

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Infer $\overline{q} \vee u$ from p and $\overline{p} \vee \overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
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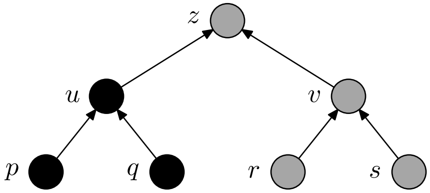

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

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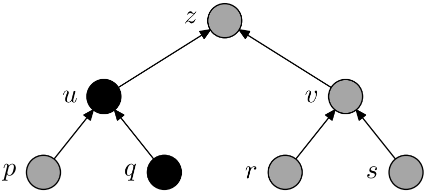

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

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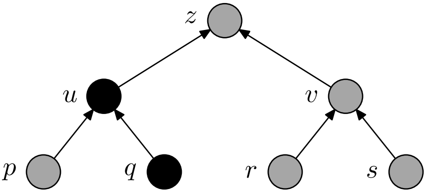

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
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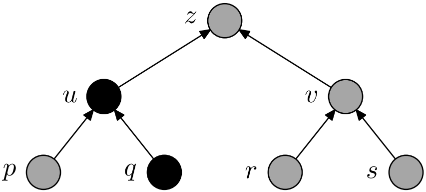

q
$\overline{q} \lor u$

Erase clause $\overline{p} \lor \overline{q} \lor u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
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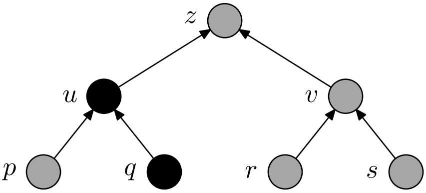

q
$\overline{q} \vee u$

Infer u from q and $\overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
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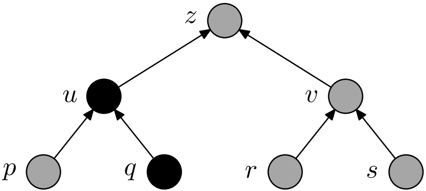
8. $\overline{z}$

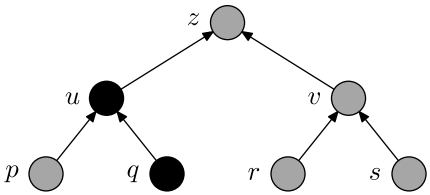


q
$\overline{q} \vee u$
u

Infer u from
q and $\overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

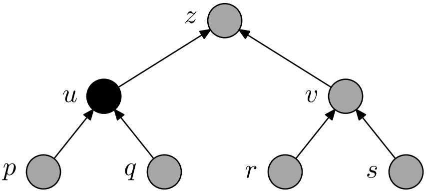


q
$\overline{q} \vee u$
u

Erase clause q

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

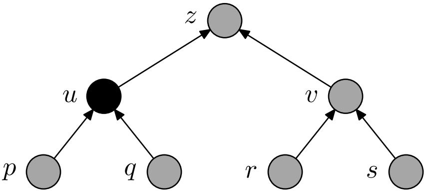


$\overline{q} \vee u$
u

Erase clause q

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

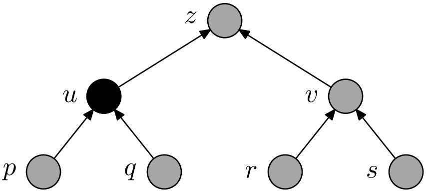


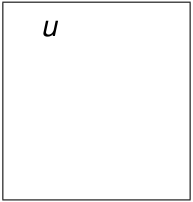
$\overline{q} \vee u$
u

Erase clause $\overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

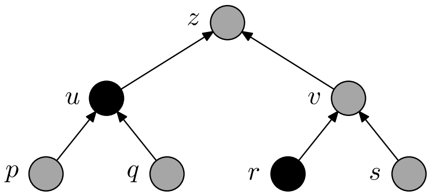




Erase clause $\overline{q} \vee u$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

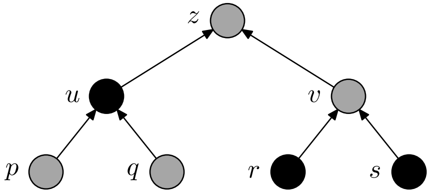


u
r

Download axiom 3: r

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

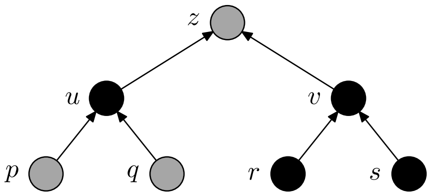


u r s

Download axiom 4: s

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

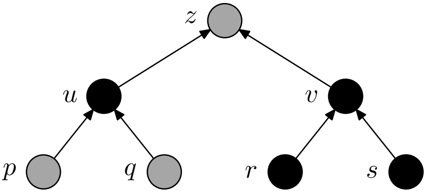


u
r
s
$\overline{r} \lor \overline{s} \lor v$

Download axiom 6: $\overline{r} \lor \overline{s} \lor v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

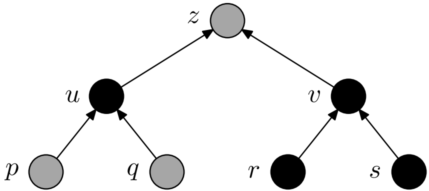


u
r
s
$\overline{r} \vee \overline{s} \vee v$

Infer $\overline{s} \vee v$ from
r and $\overline{r} \vee \overline{s} \vee v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

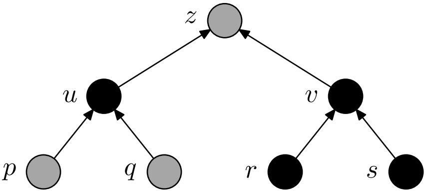


u
r
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Infer $\overline{s} \lor v$ from
r and $\overline{r} \lor \overline{s} \lor v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

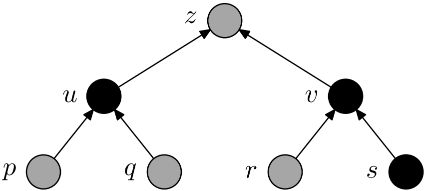


u
r
s
$\overline{r} \vee \overline{s} \vee v$
$\overline{s} \vee v$

Erase clause r

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

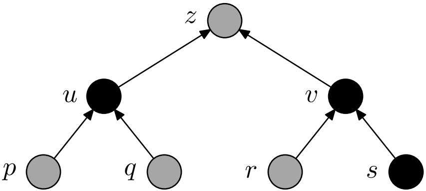


u
s
$\overline{r} \vee \overline{s} \vee v$
$\overline{s} \vee v$

Erase clause r

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

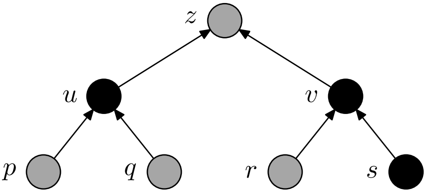


u
s
$\overline{r} \lor \overline{s} \lor v$
$\overline{s} \lor v$

Erase clause $\overline{r} \lor \overline{s} \lor v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

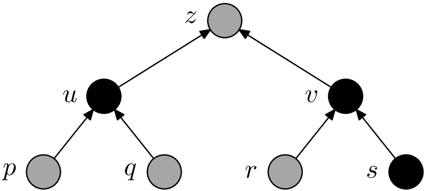


u
s
$\overline{s} \vee v$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

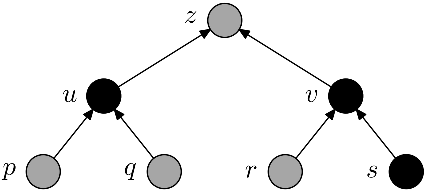


u
s
$\overline{s} \lor v$

Infer v from
s and $\overline{s} \lor v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

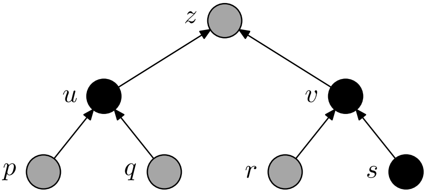


u
s
$\overline{s} \lor v$
v

Infer v from
s and $\overline{s} \lor v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

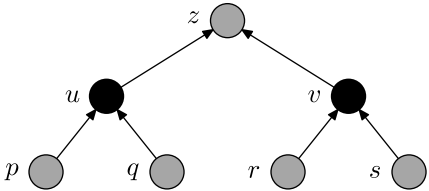


u
s
$\overline{s} \lor v$
v

Erase clause s

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

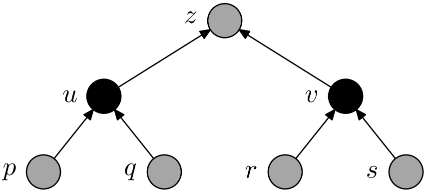


u
$\overline{s} \lor v$
v

Erase clause s

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

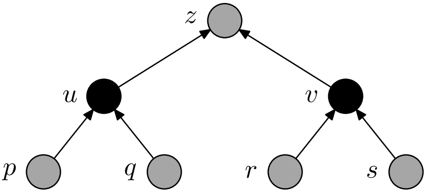


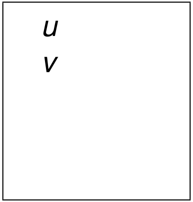
u
$\overline{s} \vee v$
v

Erase clause $\overline{s} \vee v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

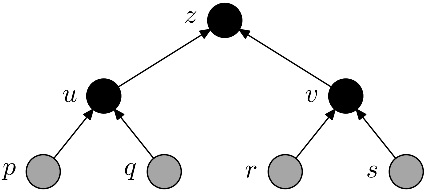




Erase clause $\overline{s} \vee v$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

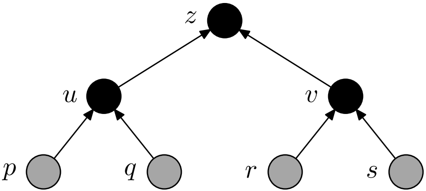


u
v
$\overline{u} \vee \overline{v} \vee z$

Download axiom 7: $\overline{u} \vee \overline{v} \vee z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

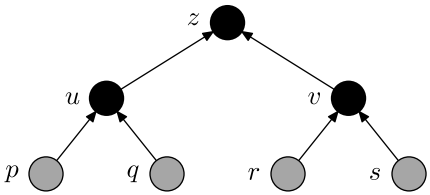


u
v
$\overline{u} \lor \overline{v} \lor z$

Infer $\overline{v} \lor z$ from
u and $\overline{u} \lor \overline{v} \lor z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

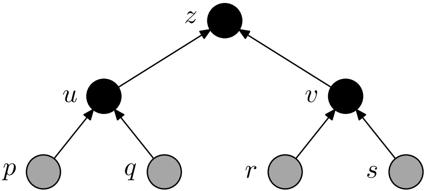


u
v
$\overline{u} \lor \overline{v} \lor z$
$\overline{v} \lor z$

Infer $\overline{v} \lor z$ from u and $\overline{u} \lor \overline{v} \lor z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

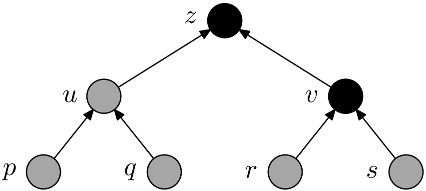


u
v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Erase clause u

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

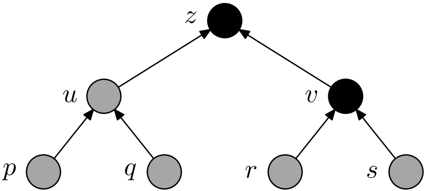


v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Erase clause u

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

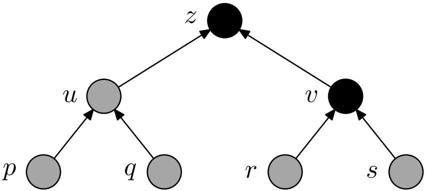


v
$\overline{u} \lor \overline{v} \lor z$
$\overline{v} \lor z$

Erase clause $\overline{u} \lor \overline{v} \lor z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



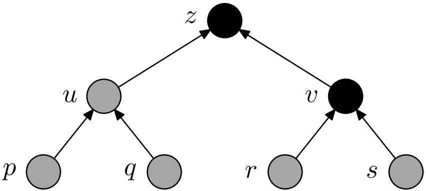
v

$\overline{v} \lor z$

Erase clause $\overline{u} \lor \overline{v} \lor z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

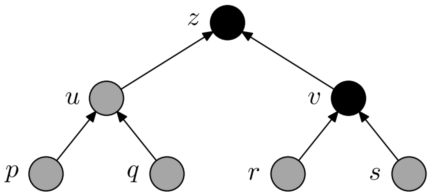


v
$\overline{v} \vee z$

Infer z from
v and $\overline{v} \vee z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

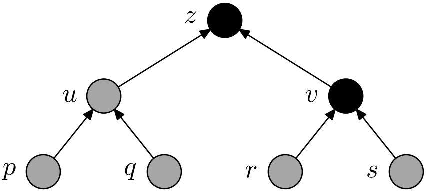


v
$\overline{v} \vee z$
z

Infer z from
v and $\overline{v} \vee z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

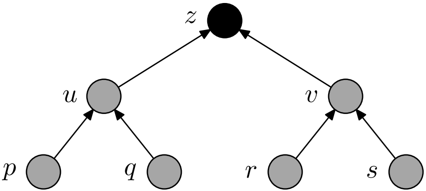


v
$\overline{v} \lor z$
z

Erase clause v

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



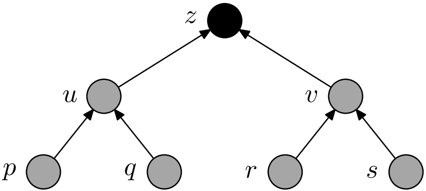
$\overline{v} \lor z$

z

Erase clause v

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



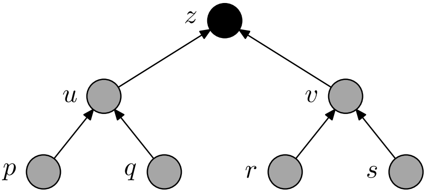
$\overline{v} \vee z$

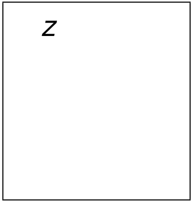
z

Erase clause $\overline{v} \vee z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

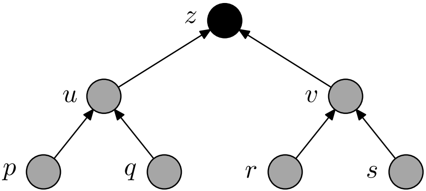




Erase clause $\overline{v} \lor z$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

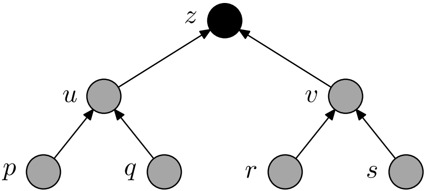


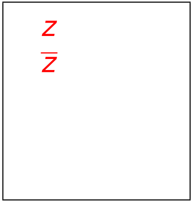
z
$\overline{z}$

Download axiom 8: $\overline{z}$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

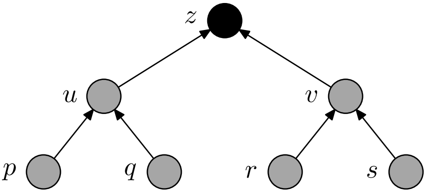




Infer 0 from
z and $\overline{z}$

Developing an Intuition for Black Pebbles

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



z
$\overline{z}$
0

Infer 0 from
z and $\overline{z}$

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \left\{\bigvee_{i=1}^{d} w_i \mid w \in W\right\} \vDash \bigvee_{i=1}^{d} v_i$.

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \{\bigvee_{i=1}^{d} w_i \mid w \in W\} \vDash \bigvee_{i=1}^{d} v_i$.

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

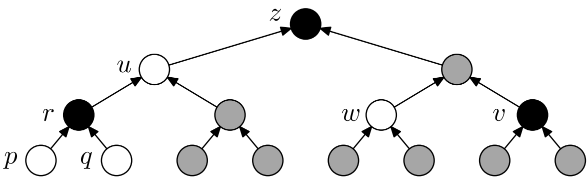
$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \left\{\bigvee_{i=1}^{d} w_i \mid w \in W\right\} \vDash \bigvee_{i=1}^{d} v_i$.

Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{l|c} \overline{u}_i \lor \overline{w}_j \lor \bigvee_{l=1}^{d} z_l & \\ \overline{p}_i \lor \overline{q}_j \lor \bigvee_{l=1}^{d} r_l & 1 \leq i,j \leq d \\ \bigvee_{l=1}^{d} v_l & \end{array} \right]$$

to induce the pebbles

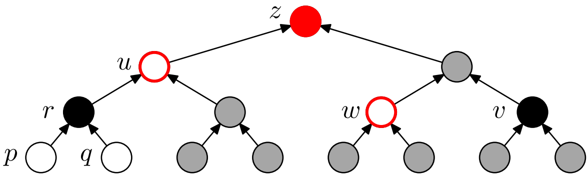


Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{l|c} \overline{u}_i \lor \overline{w}_j \lor \bigvee_{l=1}^{d} z_l & \\ \overline{p}_i \lor \overline{q}_j \lor \bigvee_{l=1}^{d} r_l & 1 \leq i,j \leq d \\ \bigvee_{l=1}^{d} v_l & \end{array} \right]$$

to induce the pebbles

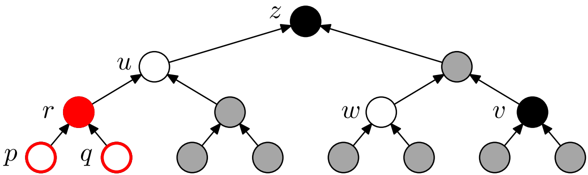


Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{l|c} \overline{u}_i \vee \overline{w}_j \vee \bigvee_{l=1}^{d} z_l & \\ \overline{p}_i \vee \overline{q}_j \vee \bigvee_{l=1}^{d} r_l & 1 \leq i,j \leq d \\ \bigvee_{l=1}^{d} v_l & \end{array} \right]$$

to induce the pebbles

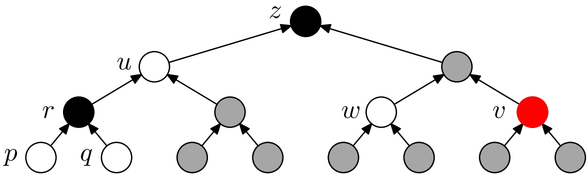


Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{l|l} \overline{u}_i \vee \overline{w}_j \vee \bigvee_{l=1}^{d} z_l & \\ \overline{p}_i \vee \overline{q}_j \vee \bigvee_{l=1}^{d} r_l & 1 \leq i,j \leq d \\ \bigvee_{l=1}^{d} v_l & \end{array} \right]$$

to induce the pebbles



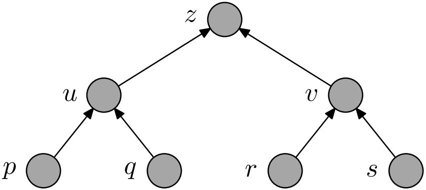
Induced Pebbles and Clause Configuration Size

- Formalizing this yields interpretation of clause configuration $\mathbb{C}_t$ derived from Peb^d_G in terms of pebbles on G
- Hope that resolution proof will correspond to black-white pebbling of G under this interpretation
- But to get lower bound on space from this we need to show that

$$\mathbb{C}_t \text{ induces many pebbles}$$
$$\Downarrow$$
$$\mathbb{C}_t \text{ contains many clauses}$$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

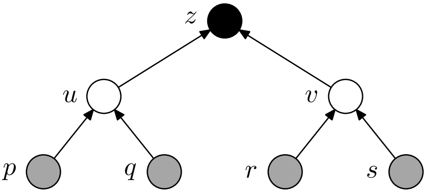
1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$



Refutation in space 3
by Ben-Sasson (2002)

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

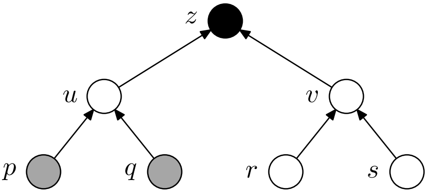


$\overline{u} \vee \overline{v} \vee z$

Download axiom 7: $\overline{u} \vee \overline{v} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

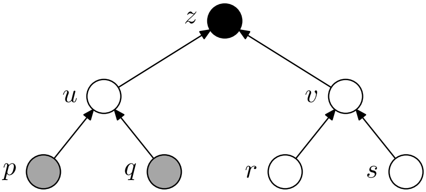


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$

Download axiom 6: $\overline{r} \vee \overline{s} \vee v$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

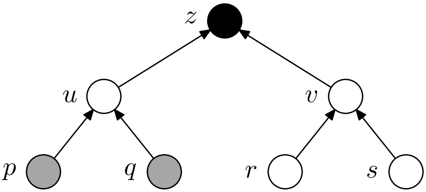


$\overline{u} \lor \overline{v} \lor z$
$\overline{r} \lor \overline{s} \lor v$

Infer $\overline{r} \lor \overline{s} \lor \overline{u} \lor z$ from $\overline{r} \lor \overline{s} \lor v$ and $\overline{u} \lor \overline{v} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

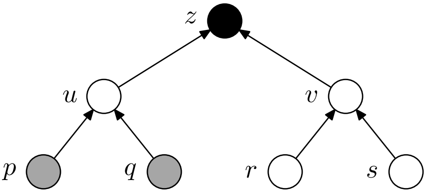


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Infer $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$ from $\overline{r} \vee \overline{s} \vee v$ and $\overline{u} \vee \overline{v} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

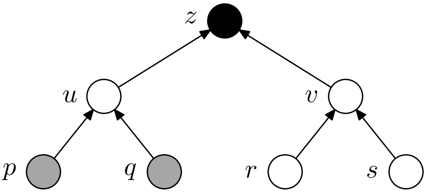


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

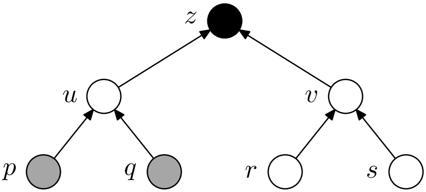


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

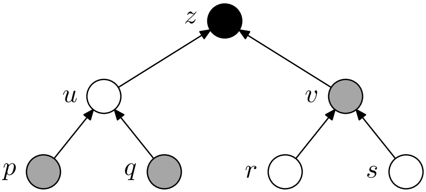


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

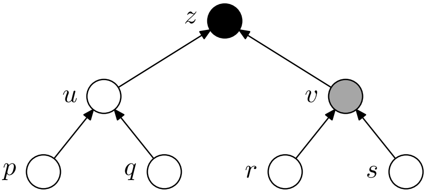


$\overline{r} \lor \overline{s} \lor \overline{u} \lor z$

Erase clause $\overline{u} \lor \overline{v} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

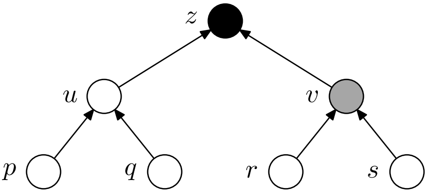


$\overline{r} \lor \overline{s} \lor \overline{u} \lor z$
$\overline{p} \lor \overline{q} \lor u$

Download axiom 5: $\overline{p} \lor \overline{q} \lor u$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

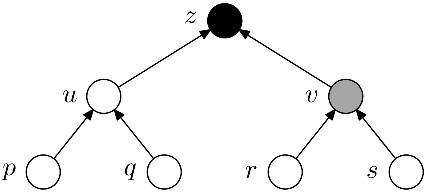


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$

Infer $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from $\overline{p} \vee \overline{q} \vee u$ and $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

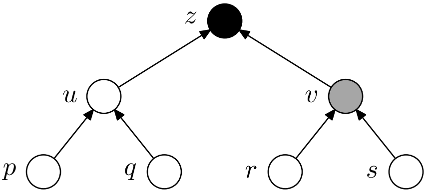


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Infer $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from $\overline{p} \vee \overline{q} \vee u$ and $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

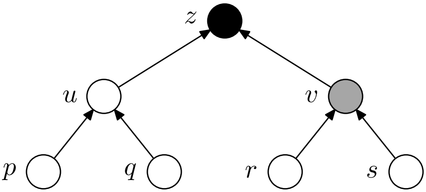


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

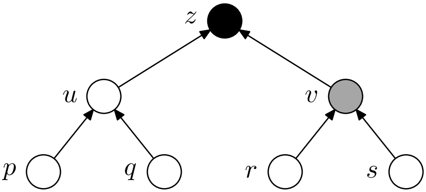


$\overline{r} \lor \overline{s} \lor \overline{u} \lor z$
$\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Erase clause $\overline{p} \lor \overline{q} \lor u$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

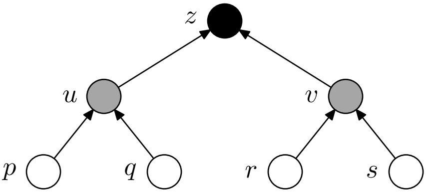


$\overline{r} \lor \overline{s} \lor \overline{u} \lor z$
$\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Erase clause $\overline{r} \lor \overline{s} \lor \overline{u} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

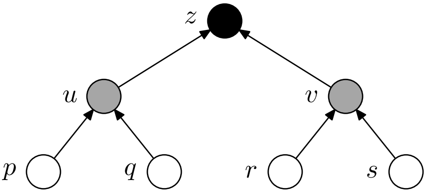


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

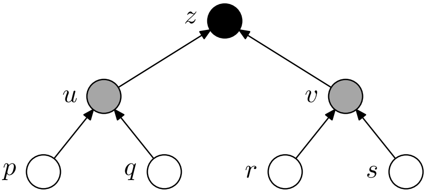


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
p

Download axiom 1: p

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

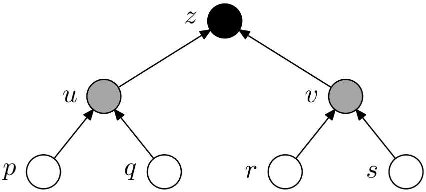


$\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$
p

Infer $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$ from p and $\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

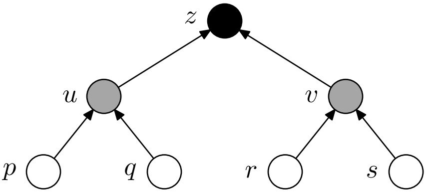


$\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$
p
$\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Infer $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$ from p and $\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

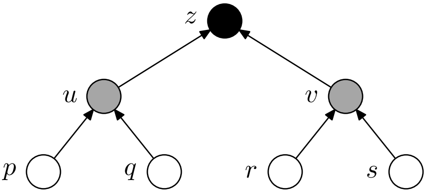


$\overline{p} \lor \overline{q} \lor \overline{r} \lor \overline{s} \lor z$
p
$\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Erase clause p

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

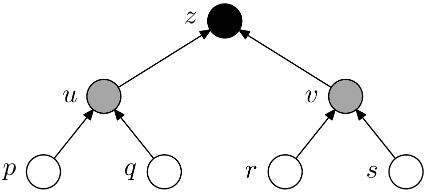


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause p

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

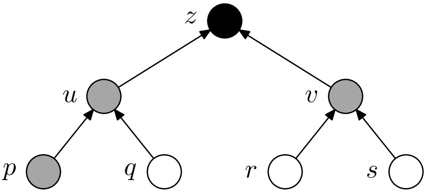


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

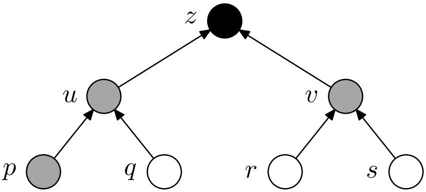


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

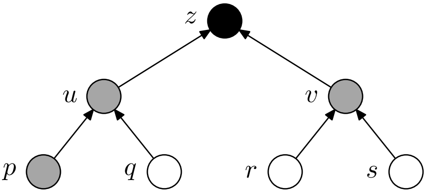


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q

Download axiom 2: q

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

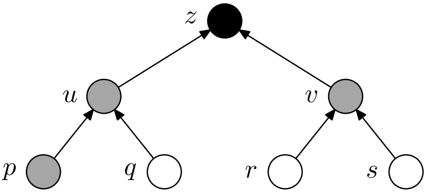


$\overline{q} \lor \overline{r} \lor \overline{s} \lor z$
q

Infer $\overline{r} \lor \overline{s} \lor z$ from q and $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

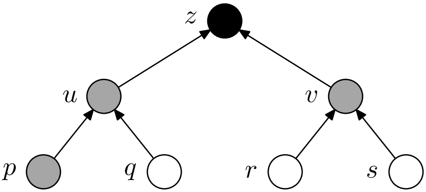


$\overline{q} \lor \overline{r} \lor \overline{s} \lor z$
q
$\overline{r} \lor \overline{s} \lor z$

Infer $\overline{r} \lor \overline{s} \lor z$ from q and $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

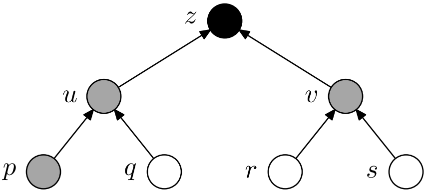


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q
$\overline{r} \vee \overline{s} \vee z$

Erase clause q

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

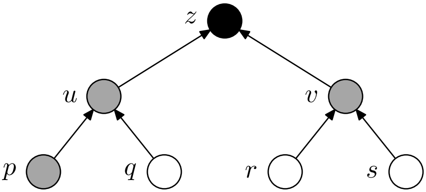


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{r} \vee \overline{s} \vee z$

Erase clause q

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

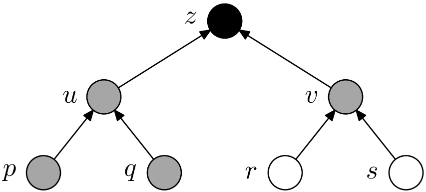


$\overline{q} \lor \overline{r} \lor \overline{s} \lor z$
$\overline{r} \lor \overline{s} \lor z$

Erase clause $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

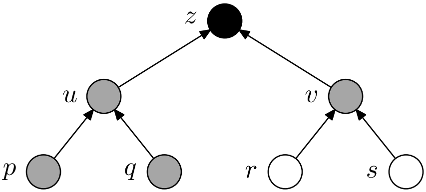


$\overline{r} \lor \overline{s} \lor z$

Erase clause $\overline{q} \lor \overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

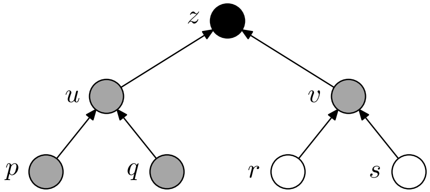


$\overline{r} \vee \overline{s} \vee z$
r

Download axiom 3: r

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

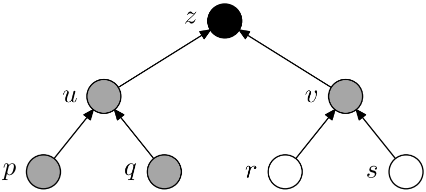


$\overline{r} \vee \overline{s} \vee z$
r

Infer $\overline{s} \vee z$ from r and $\overline{r} \vee \overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

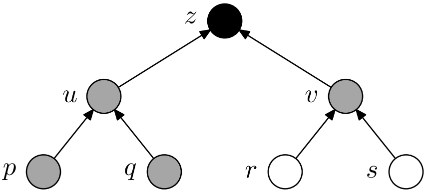


$\overline{r} \lor \overline{s} \lor z$
r
$\overline{s} \lor z$

Infer $\overline{s} \lor z$ from
r and $\overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

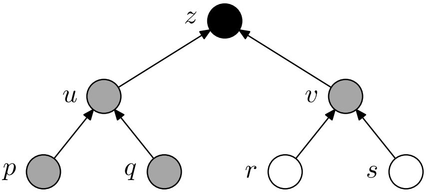


$\overline{r} \vee \overline{s} \vee z$
r
$\overline{s} \vee z$

Erase clause r

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

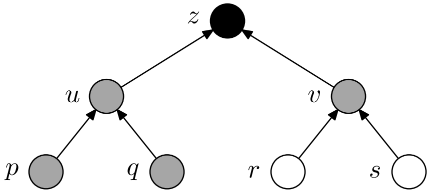


$\overline{r} \vee \overline{s} \vee z$
$\overline{s} \vee z$

Erase clause r

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

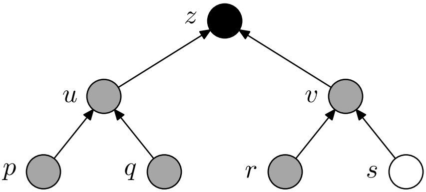


$\overline{r} \lor \overline{s} \lor z$
$\overline{s} \lor z$

Erase clause $\overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \lor \overline{q} \lor u$
6. $\overline{r} \lor \overline{s} \lor v$
7. $\overline{u} \lor \overline{v} \lor z$
8. $\overline{z}$

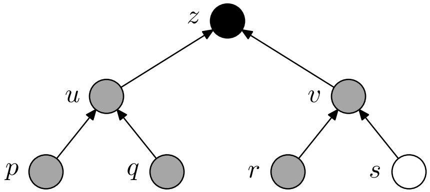


$\overline{s} \lor z$

Erase clause $\overline{r} \lor \overline{s} \lor z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

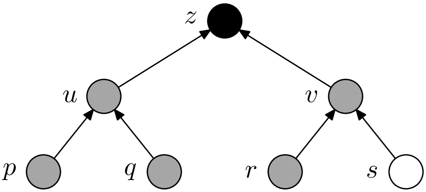


$\overline{s} \vee z$
s

Download axiom 4: s

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

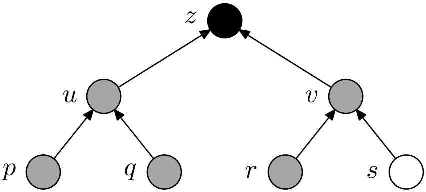


$\overline{s} \vee z$
s

Infer z from
s and $\overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

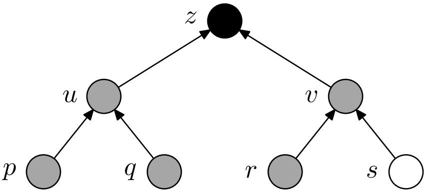


$\overline{s} \vee z$
s
z

Infer z from s and $\overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

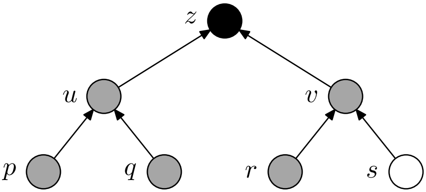


$\overline{s} \vee z$
s
z

Erase clause s

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

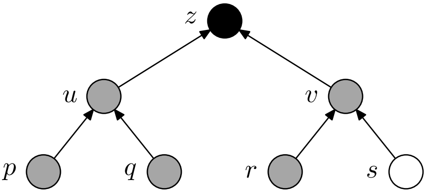


$\overline{s} \vee z$
z

Erase clause s

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

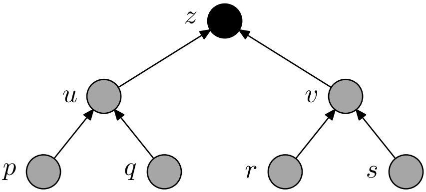


$\overline{s} \vee z$
z

Erase clause $\overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

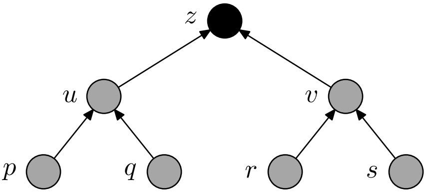


z

Erase clause $\overline{s} \vee z$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

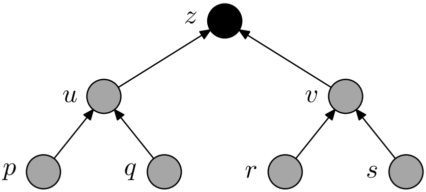


z
$\overline{z}$

Download axiom 8: $\overline{z}$

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

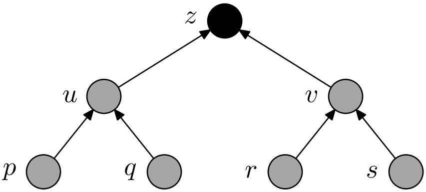


z
$\overline{z}$

Infer 0 from $\overline{z}$ and z

Many Pebbles $\not\Rightarrow$ Many Clauses for 1 Variable / Vertex!

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



z
$\overline{z}$
0

Infer 0 from $\overline{z}$ and z

But # Pebbles $\geq$ # Clauses for $d > 1$

This “top-down” proof in space 3 generalizes to any DAG G

- In terms of our induced pebble configurations:
  white pebbles are free for $d = 1$!
- In a sense, this is exactly why $Sp\left(Peb^1_G \vdash 0\right) = \mathcal{O}(1)$
- But for $d > 1$ variables per vertex we can prove that
  # clauses $\geq$ # induced pebbles

Induced Pebbles Break The Pebbling Rules

Unfortunately, our interpretation of resolution refutations does not yield "well-behaved" pebblings

- Erasures can (and will) lead to large blocks of black and white pebbles suddenly just disappearing
- Need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex—label each black pebble with the white pebbles it depends on
- "Illegal" removal of white pebble from w OK provided that all black pebbles labelled with w are removed as well!
- Also white pebbles may slide upwards and black pebbles slide downwards—"backward" reversal moves

If you can't beat 'em, join 'em! Define new labelled pebble game

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If you can't beat 'em, join 'em! Define new labelled pebble game

Induced Pebbles Break The Pebbling Rules

Unfortunately, our interpretation of resolution refutations does not yield "well-behaved" pebblings

- Erasures can (and will) lead to large blocks of black and white pebbles suddenly just disappearing
- Need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex—label each black pebble with the white pebbles it depends on
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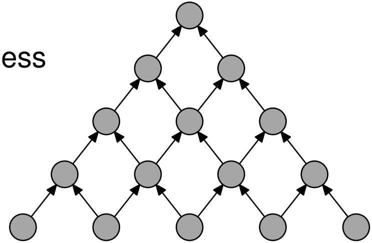

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For T_h a complete binary tree of height h, the labelled pebbling price is $\text{L-Peb}(T_h) = \Theta(\text{BW-Peb}(T_h)) = \Theta(h)$.

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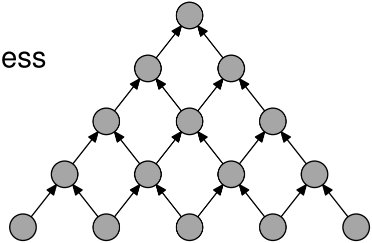

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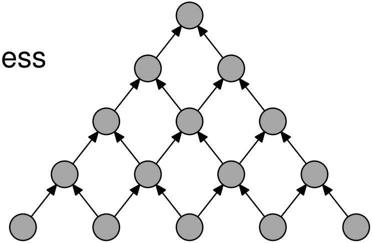

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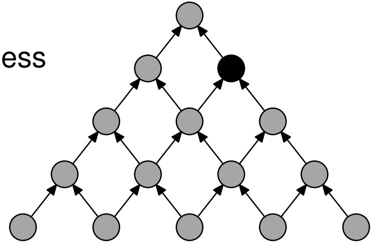

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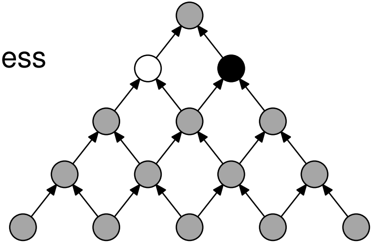

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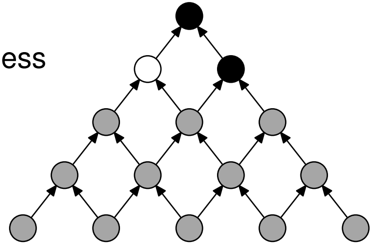

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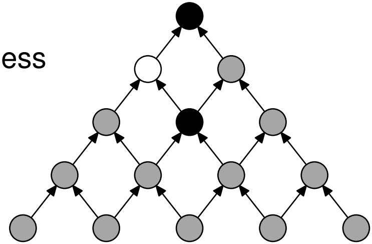

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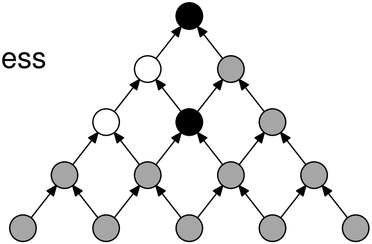

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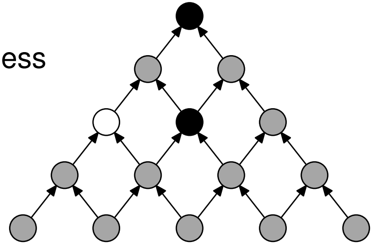

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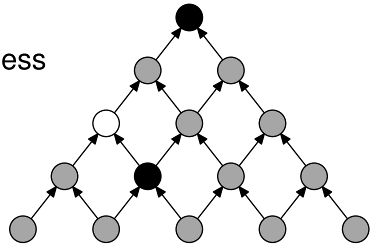

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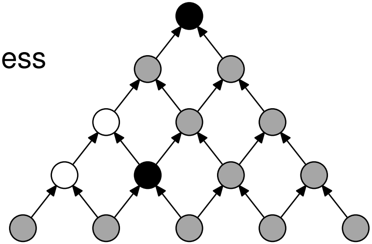

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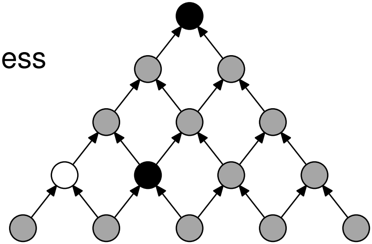

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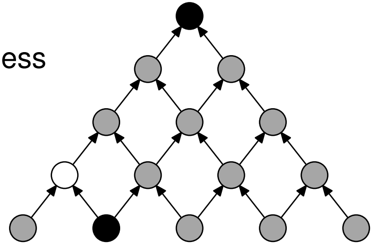

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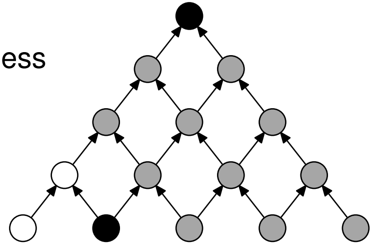

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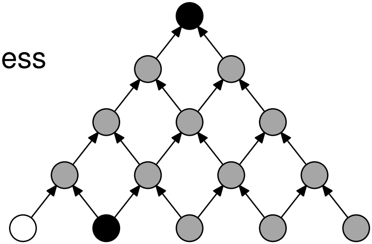

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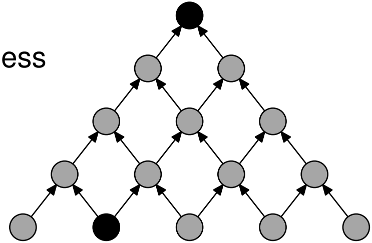

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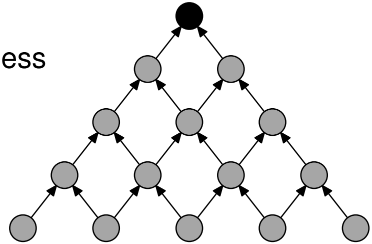

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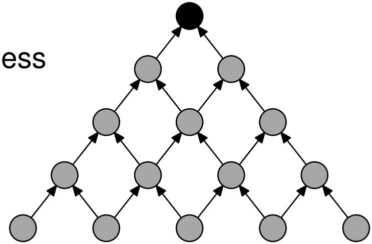

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Main Theorem

Theorem

Space of refuting pebbling contradiction of degree $d \geq 2$ over complete binary tree of height h is $Sp\left(Peb^{d}_{T_h} \vdash 0\right) = \Theta(h)$

Proof sketch.

- Upper bound easy (use "black-pebbling" resolution proof)
- For lower bound, let $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ refutation of $Peb^{d}_{T_h}$
- π induces labelled pebbling of $T_h \Rightarrow \exists$ some $\mathbb{C}_t \in \pi$ corresponding to $\Omega(h)$ pebbles
- # pebbles in T_h at time $t \leq$ # clauses in $\mathbb{C}_t$ (since $d \geq 2$)
- Thus $Sp(\pi) \geq |\mathbb{C}_t| \geq$ # pebbles induced by $\mathbb{C}_t = \Omega(h)$ □

A Separation of Space and Width in Resolution

Corollary

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof.

Know $W\left(Peb_G^d \vdash 0\right) = \mathcal{O}(d)$ for all G

Peb_G^d is $(2+d)$-CNF formula

Fix $d \geq 2$, let $F_n = Peb_{T_h}^d$ for $h = \lfloor \log(n+1) \rfloor$ and use Main Theorem □

References for Space-Width Separation

Published as *Narrow Proofs May Be Spacious: Separating Space and Width in Resolution*

Extended abstract in STOC '06: all formal definitions + statements of theorems with proof sketches

Full-length paper with all technical details to appear in SIAM Journal on Computing—drop me a line to get current version

Lower Bounds on Variable Space?

Open Question

Is there a CNF formula family $\{F_n\}_{n=1}^{\infty}$ *of size* $\mathcal{O}(n)$ *such that* $\mathit{VarSp}(F_n \vdash 0) = \Omega(n^2)$*?*
Or at least such that $\mathit{VarSp}(F_n \vdash 0) = \omega(n)$*?*

Mentioned in Aleknovich et al. (2000)

Answer conjectured to be “yes”

Still open as far as I know

Length-Width Trade-offs: Are Short Proofs Narrow?

Ben-Sasson & Wigderson (1999) showed that given refutation in length L, can find refutation in width $\mathcal{O}(\sqrt{n \log L})$

But not the same refutation! Exponential blow-up in length!

Is this increase in length necessary?

Open Question

Given refutation of k-CNF formula F over n variables in length $L(\pi) = L$, is there a refutation π' in width $W(\pi') = \mathcal{O}(\sqrt{n \log L})$ with length no more than, say, $L(\pi') = \mathcal{O}(L)$ or at most $poly(L)$?

Or can we find formula family with length-width trade-off?

Length-Space Trade-offs: Are Tight Proofs Short?

Given refutation in small space $\Rightarrow$ exists refutation in short length (by Atserias & Dalmau 2003)

But again not the same refutation

For concreteness, fix space to constant:

Open Question

Given polynomial-size k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ *with* $Sp(F_n \vdash 0) = \mathcal{O}(1)$*, is there a refutation* π' *with* $L(\pi') = poly(n)$ *and* $Sp(\pi') = \mathcal{O}(1)$*?*

Or can it be that restricting the space, we end up with really long refutations? (Compare Hertel & Pitassi 2007)

Length-Space Trade-offs: Are Short Proofs Tight?

Recall: ∃ short refutation ⇒ ∃ narrow refutation

Is it true that ∃ short refutation ⇒ ∃ small space refutation?

Or can short refutations be arbitrarily complex w.r.t. space?

My Conjecture

Exists family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ such that $L(F_n \vdash 0) = \mathcal{O}(n)$ but $Sp(F_n \vdash 0) = \Omega(n/\log n)$

Would separate length and space in strongest sense possible (given length n, space $\mathcal{O}(n/\log n)$ always possible)

Could be really bad news for proof search algorithms

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Plausible Candidate: Pebbling Contradictions

Pebbling contradictions refutable in linear length

For binary trees, space grows like $\mathit{BW\text{-}Peb}(T_h)$

Intuition

For any DAG G, from resolution refutation of pebbling contradiction should be possible to extract black-white pebbling

Sufficient! $\exists \{G_n\}$ of size $\mathcal{O}(n)$ with $\mathit{BW\text{-}Peb}(G_n) = \Theta(n/\log n)$

Problem

What if a refutation doesn't feel like respecting our intuition?

E.g. might have derived if u and v true, then x or y must be true for distinct vertices u, v, x, y

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Interpret Refutations as "Multi-Pebblings"

Suggested solution: introduce "fuzzy" black pebbles covering multiple vertices

Notation $[B]\langle W \rangle$ for

- black "multi-pebble" B with
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Require $B \cap W = \emptyset$

Introduction move:
Black pebble on v with white pebbles on predecessors

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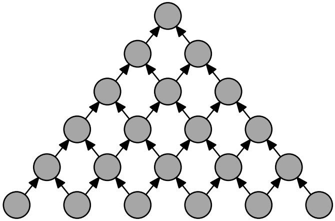

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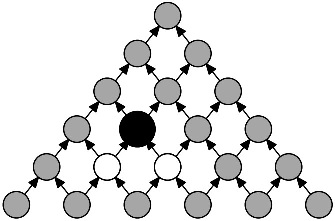
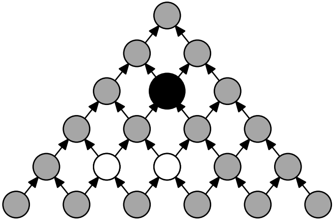


Inflations and Merger Moves

Inflation move:
Enlarge black multi-pebble and/or add white pebbles



Merger move:
Join $[B_1]\langle W_1\rangle$ & $[B_2]\langle W_2\rangle$ by removing unique common black-white vertex

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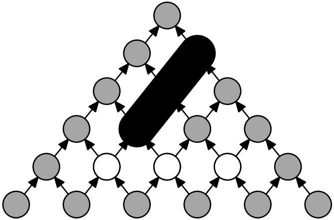

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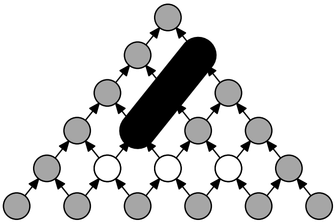

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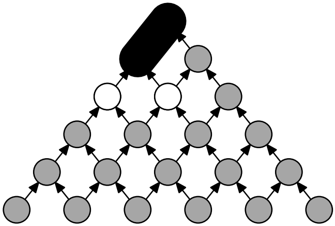
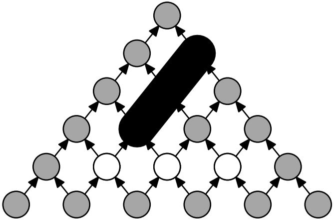
common black-white vertex



Inflations and Merger Moves

Inflation move:
Enlarge black multi-pebble and/or add white pebbles



Merger move:
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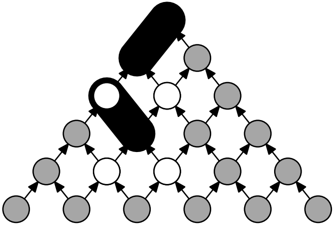

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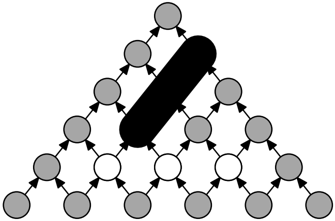

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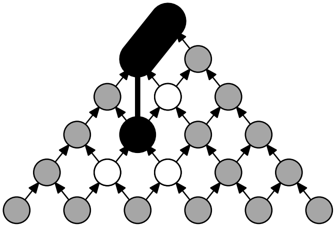

The Multi-Pebble Game in All Its Formal Glory

Multi-pebble game

Multi-pebbling of G: sequence of sets $\mathcal{M} = \{\mathbb{M}_0, \ldots, \mathbb{M}_\tau\}$ such that $\mathbb{M}_0 = \emptyset$, $\mathbb{M}_\tau = \{[z]\langle\emptyset\rangle\}$ and $\mathbb{M}_t$ is obtained from $\mathbb{M}_{t-1}$ by:

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Multi-Pebbling Price

Multi-pebbling price

- Charge for every white pebble
- Charge for longest sequence of black multi-pebbles $B_1, \ldots, B_s$ such that $B_i \not\subseteq \bigcup_{j<i} B_j$

M-Peb(G) = min cost to get to $[z]\langle\emptyset\rangle$ for any pebbling of G

Example: these overlapping pebbles have cost 5

- white pebbles cost 3
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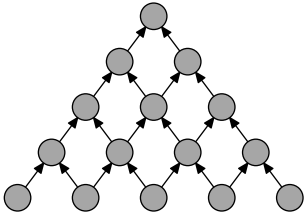

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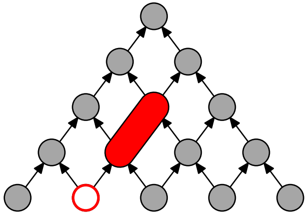

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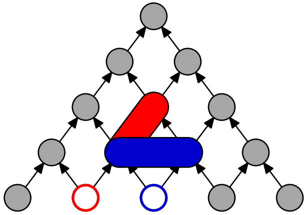

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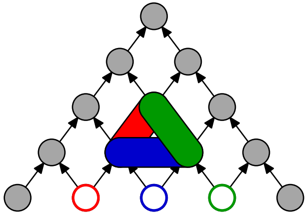

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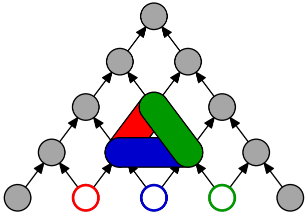

Multi-Pebbling and Space Lower Bounds

Resolution refutations correspond to multi-pebblings

Lower bounds on *M-Peb*(G)

$\Downarrow$

separation of length and space

Sufficient: show inflation can't help

$\Downarrow$

M-Peb$(G) \geq$ *BW-Peb*(G)

Take-Home Message

- Lots of nice (and surprising!) results relating length, width and space
- Quite a few nice open problems left
- Why not start by attacking the multi-pebble game? ☺