

Narrow Proofs May Be Spacious: Separating Space and Width in Resolution

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Executive Summary of Talk (1 / 2)

Resolution: proof system for refuting CNF formulas

Perhaps *the* most studied system in proof complexity

Also used in many real-world automated theorem provers

- Haken (1985): exponential lower bound on **proof length** (# clauses in a resolution proof)
- Ben-Sasson & Wigderson (1999): lower bound on length in terms of **proof width** (size of largest clause in proof)
- Results on width lead to question whether other complexity measures could yield interesting insights as well

Executive Summary of Talk (2 / 2)

- Esteban & Torán (1999): **proof space**
(maximal # clauses in memory while verifying proof)
- Many lower bounds for space proven
All turned out to match width bounds!
Coincidence?
- Atserias & Dalmau (2003): **space** \geq **width** – **constant** for
 k -CNF formulas
- **Problem left open**: Do space and width coincide or not?

We resolve this question: separation of space and width

Outline

- 1 Background
 - Preliminaries
 - Overview of Previous Work
- 2 Pebble Games and Resolution
 - Pebble Games
 - Pebbling Contradictions
 - Resolution Refutations of Pebbling Contradictions
- 3 A Separation of Space and Width
 - Interpreting Clauses as Pebbles
 - Many Pebbles Imply Many Clauses
 - The Induced Black-White Pebble Game
 - Putting It All Together
- 4 Conclusion and Open Problems

Some Notation and Terminology

- **Literal** a : variable x or its negation \bar{x}
- **Clause** $C = a_1 \vee \dots \vee a_k$: set of literals
At most k literals: **k -clause**
- **CNF formula** $F = C_1 \wedge \dots \wedge C_m$: set of clauses
 k -CNF formula: CNF formula consisting of k -clauses
(assume k fixed)

Some More Notation and Terminology

- $Vars(\cdot)$: set of variables in clause or formula
- $Lit(\cdot)$: set of literals in clause or formula
- Truth value assignment α makes
 - clause true if one literal true
 - CNF formula true if all clauses true
- $F \models D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \dots, n\}$

Resolution Rule

Resolution rule:

$$\frac{B \vee x \quad C \vee \bar{x}}{B \vee C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

Prove F **unsatisfiable** by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

Resolution Derivation

Sequence of sets of clauses, or **clause configurations**,
 $\{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and \mathbb{C}_t follows from \mathbb{C}_{t-1} by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (**axiom**)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{B \vee C\}$ for clause $B \vee C$ inferred by
resolution rule from $B \vee x, C \vee \bar{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F :

Derivation $\{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

Resolution refutation of F :

Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Derivation Length, Width and Space

- **Length** $L(\pi)$ of derivation $\pi : F \vdash D$
distinct clauses in all of π
- **Width** $W(\pi)$ of derivation $\pi : F \vdash D$
literals in largest clause in π
- **Space** $Sp(\pi)$ of derivation $\pi : F \vdash D$
clauses in largest clause configuration $\mathbb{C}_t \in \pi$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	0
Width so far	0
Space so far	0

$$\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

Empty start configuration

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	1
Width so far	1
Space so far	1

$$\left[\begin{array}{c} p_1 \\ \end{array} \right]$$

Download axiom 1: p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	2
Width so far	1
Space so far	2

$$\left[\begin{array}{l} p_1 \\ q_1 \end{array} \right]$$

Download axiom 2: q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	3
Width so far	3
Space so far	3

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Download axiom 5: $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	3
Width so far	3
Space so far	3

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee u_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee u_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer u_1 from
 q_1 and $\bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	5
Width so far	3
Space so far	4

$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \\ u_1 \end{array} \right]$$

Infer u_1 from
 q_1 and $\bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	5
Width so far	3
Space so far	4

$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \\ u_1 \end{array} \right]$$

Erase clause $\bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	5
Width so far	3
Space so far	4

$$\left[\begin{array}{c} q_1 \\ u_1 \end{array} \right]$$

Erase clause $\bar{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	5
Width so far	3
Space so far	4

$$\left[\begin{array}{c} q_1 \\ u_1 \end{array} \right]$$

Erase clause q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	5
Width so far	3
Space so far	4

$$\left[\begin{array}{c} u_1 \\ \\ \\ \end{array} \right]$$

Erase clause q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	6
Width so far	3
Space so far	4

$$\left[\begin{array}{c} u_1 \\ r_1 \end{array} \right]$$

Download axiom 3: r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	7
Width so far	3
Space so far	4

$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \end{array} \right]$$

Download axiom 4: s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	8
Width so far	3
Space so far	4

$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Download axiom 6: $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	8
Width so far	3
Space so far	4

$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee v_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee v_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer v_1 from
 s_1 and $\bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	10
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \\ v_1 \end{array} \right]$$

Infer v_1 from
 s_1 and $\bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	10
Width so far	3
Space so far	5

$$\left[\begin{array}{c} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \\ v_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	10
Width so far	3
Space so far	5

$$\left[\begin{array}{c} u_1 \\ s_1 \\ v_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	10
Width so far	3
Space so far	5

$$\left[\begin{array}{c} u_1 \\ s_1 \\ v_1 \end{array} \right]$$

Erase clause s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	10
Width so far	3
Space so far	5

$$\left[\begin{array}{c} u_1 \\ v_1 \end{array} \right]$$

Erase clause s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	11
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \end{array} \right]$$

Download axiom 7: $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	11
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{v}_1 \vee z_1$ from
 u_1 and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{v}_1 \vee z_1$ from
 u_1 and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause u_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause u_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer z_1 from
 v_1 and $\bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Infer z_1 from
 v_1 and $\bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Erase clause $\bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{c} v_1 \\ z_1 \end{array} \right]$$

Erase clause $\bar{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{c} v_1 \\ z_1 \end{array} \right]$$

Erase clause v_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{c} z_1 \\ \end{array} \right]$$

Erase clause v_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	14
Width so far	3
Space so far	5

$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Download axiom 8: \bar{z}_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	14
Width so far	3
Space so far	5

$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Infer 0 from
 z_1 and \bar{z}_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$
6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$
7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$
8. \bar{z}_1

Length so far	15
Width so far	3
Space so far	5

$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \\ 0 \end{array} \right]$$

Infer 0 from
 z_1 and \bar{z}_1

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi: F \vdash 0} \{L(\pi)\}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi: F \vdash 0} \{W(\pi)\}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi: F \vdash 0} \{Sp(\pi)\}$$

$$L(F \vdash 0) \leq 2^{(\# \text{ variables in } F + 1)}$$

$$W(F \vdash 0) \leq \# \text{ variables in } F$$

$$Sp(F \vdash 0) \leq \min(\# \text{ variables in } F, \# \text{ clauses in } F) + \mathcal{O}(1)$$

Connection between Length and Width

A **narrow** resolution proof is necessarily **short**.

For a proof in width w , $(2 \cdot |\text{Vars}(F)|)^w$ is an upper bound on the number of possible clauses.

There is a **kind of converse** to this:

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a k -CNF formula F over n variables is

$$W(F \vdash 0) = \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

This bound on width in terms of length is essentially optimal (Bonnet & Galesi 1999).

Connection between Space and Width

All previously shown lower bounds on space coincide with lower bounds on width—**true in general?**

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k -CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?

Or is there a k -CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

Pebbles Games on Graphs

One-player game played on directed acyclic graphs (DAGs)

- Devised for studying programming languages and compiler construction
- Have found a variety of applications in complexity theory

Conventions

- $V(G)$ denote the vertices of a DAG G
- vertices with indegree 0 are **sources**
- vertices with outdegree 0 are **targets**

Only consider DAGs with **single target z** and **indegree 2 for all non-source vertices**

Formal Definition of Pebble Game

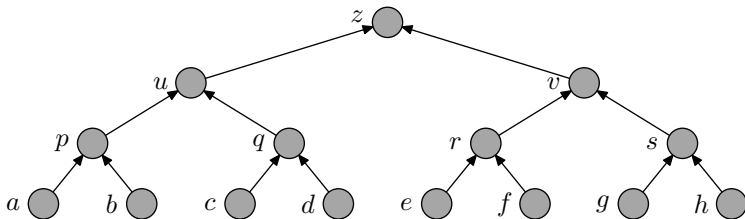
Pebble configuration: pair of subsets $\mathbb{P} = (B, W)$ of black- and white-pebbled vertices

Black-white pebbling: sequence $\mathcal{P} = \{\mathbb{P}_0, \dots, \mathbb{P}_\tau\}$ such that $\mathbb{P}_0 = (\emptyset, \emptyset)$ and \mathbb{P}_t follows from \mathbb{P}_{t-1} by one of the rules:

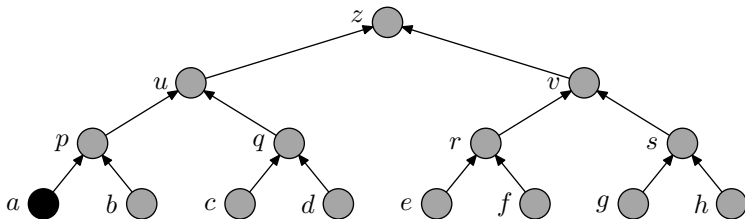
- 1 Can **place black pebble** on (empty) vertex v if all immediate predecessors have pebbles on them
- 2 Can always **remove black pebble** from vertex
- 3 Can always **place white pebble** on (empty) vertex
- 4 Can **remove white pebble** from v if all immediate predecessors have pebbles on them

Goal: reach $\mathbb{P}_\tau = (\{z\}, \emptyset)$ using few pebbles

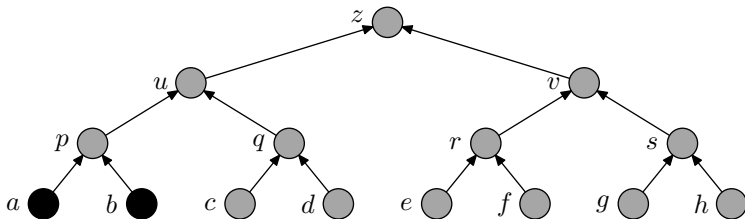
Black-White Pebbling of Binary Tree of Height 3



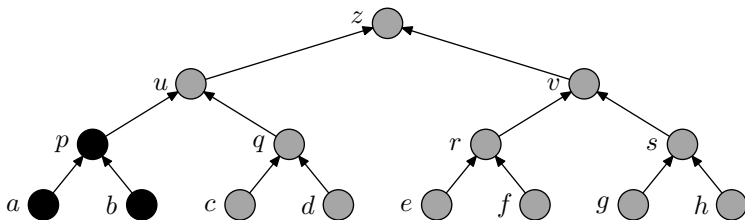
Black-White Pebbling of Binary Tree of Height 3



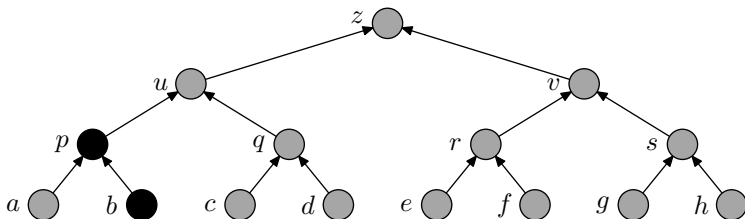
Black-White Pebbling of Binary Tree of Height 3



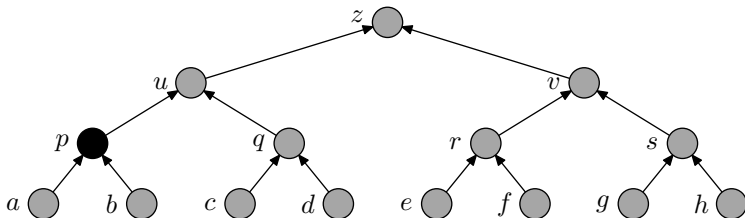
Black-White Pebbling of Binary Tree of Height 3



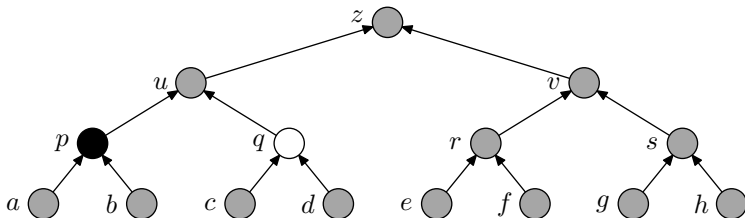
Black-White Pebbling of Binary Tree of Height 3



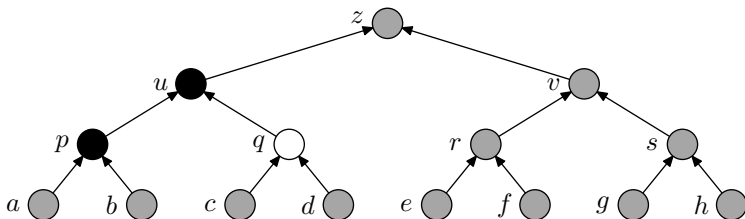
Black-White Pebbling of Binary Tree of Height 3



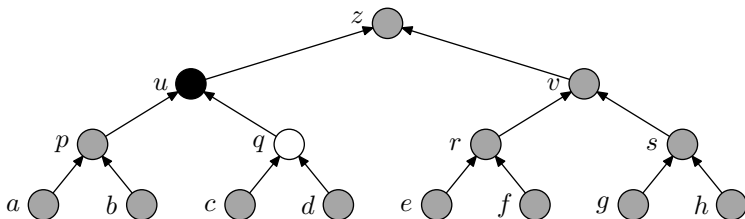
Black-White Pebbling of Binary Tree of Height 3



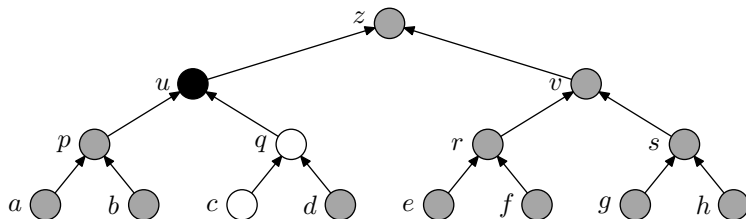
Black-White Pebbling of Binary Tree of Height 3



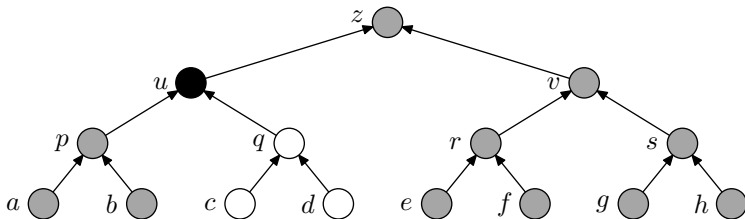
Black-White Pebbling of Binary Tree of Height 3



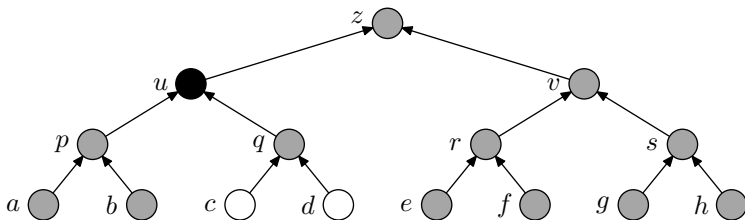
Black-White Pebbling of Binary Tree of Height 3



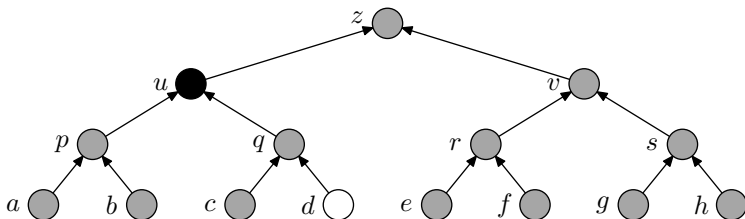
Black-White Pebbling of Binary Tree of Height 3



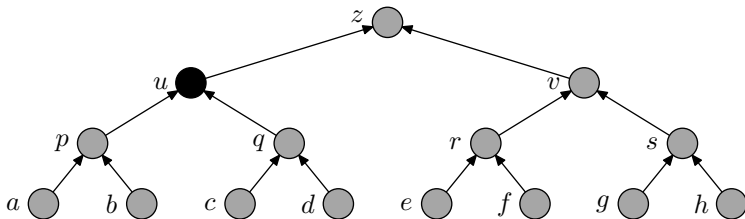
Black-White Pebbling of Binary Tree of Height 3



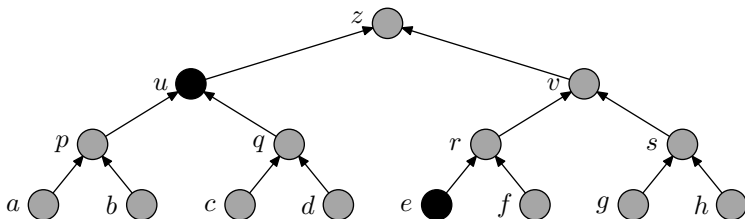
Black-White Pebbling of Binary Tree of Height 3



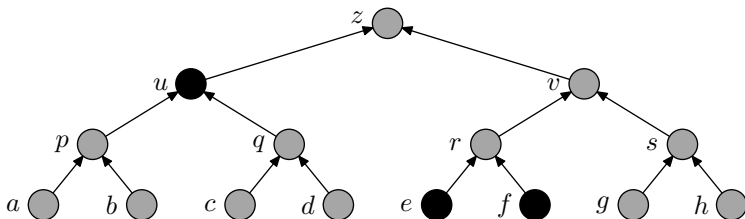
Black-White Pebbling of Binary Tree of Height 3



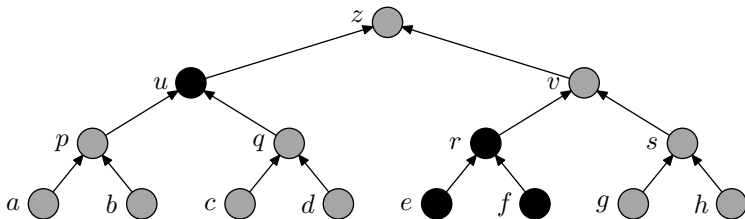
Black-White Pebbling of Binary Tree of Height 3



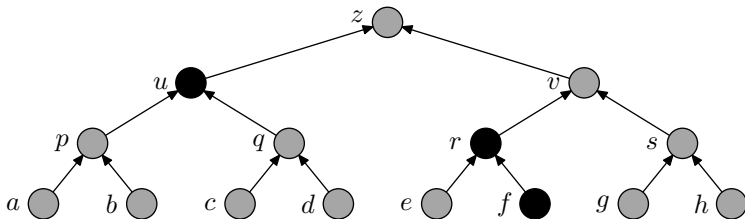
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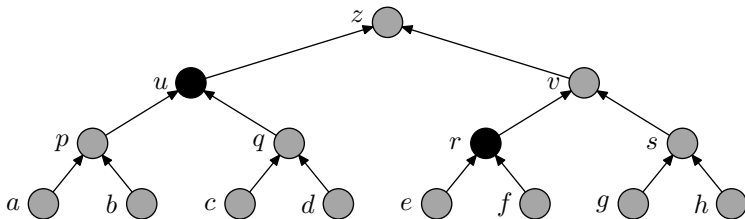
Black-White Pebbling of Binary Tree of Height 3



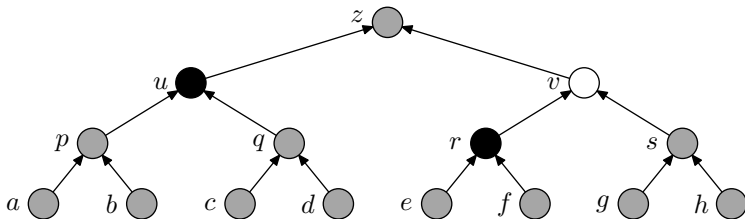
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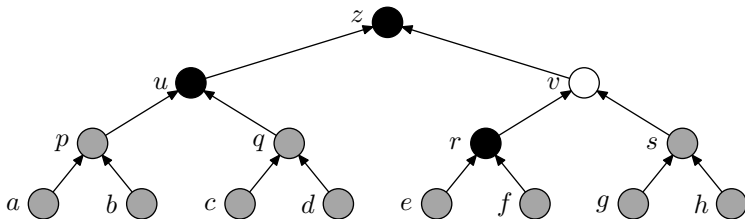
Black-White Pebbling of Binary Tree of Height 3



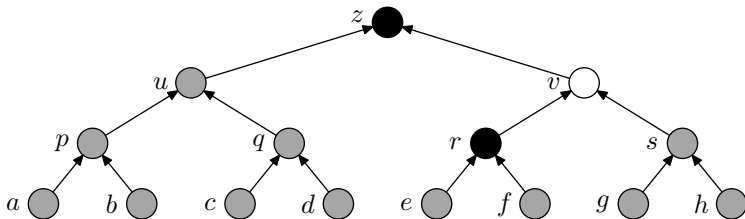
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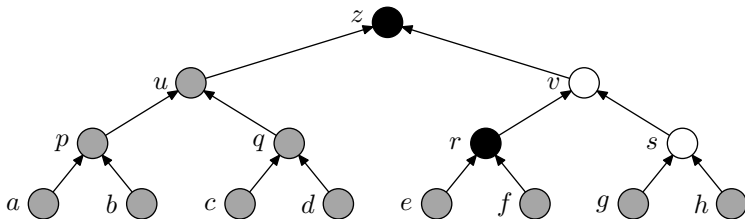
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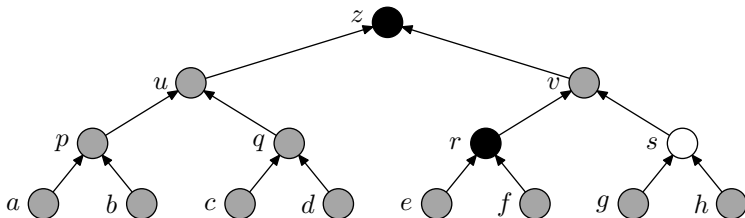
Black-White Pebbling of Binary Tree of Height 3



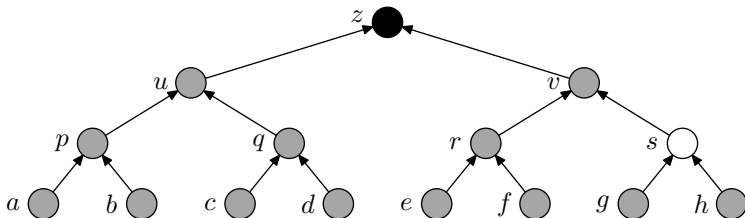
Black-White Pebbling of Binary Tree of Height 3



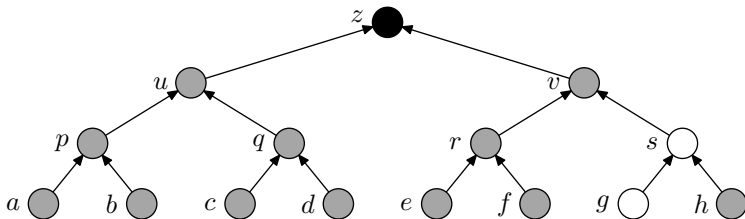
Black-White Pebbling of Binary Tree of Height 3



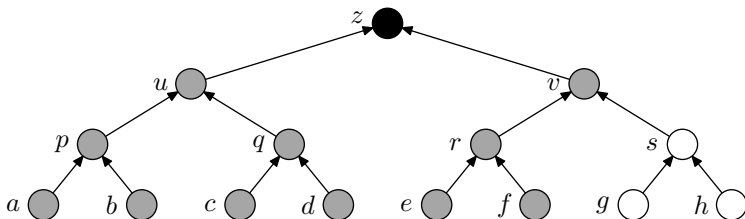
Black-White Pebbling of Binary Tree of Height 3



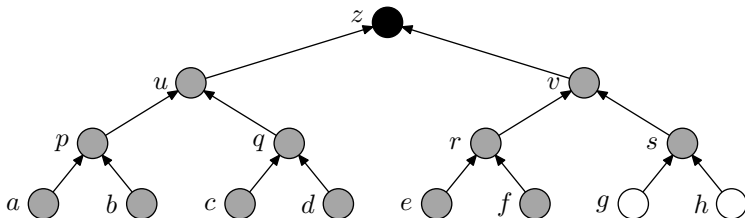
Black-White Pebbling of Binary Tree of Height 3



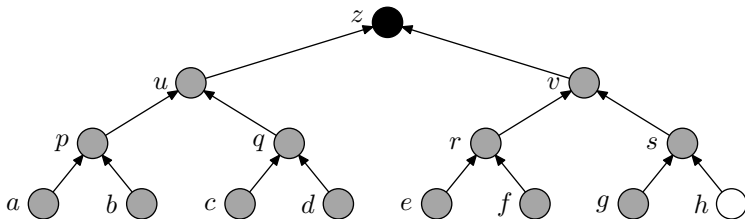
Black-White Pebbling of Binary Tree of Height 3



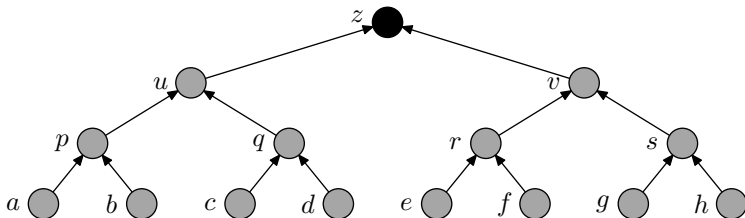
Black-White Pebbling of Binary Tree of Height 3



Black-White Pebbling of Binary Tree of Height 3



Black-White Pebbling of Binary Tree of Height 3



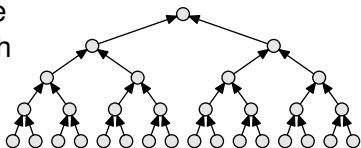
Pebbling Price

- Cost of pebbling $\mathcal{P} = \{\mathbb{P}_0, \dots, \mathbb{P}_\tau\}$:
max # pebbles in any $\mathbb{P}_t = (B_t, W_t)$
- **Black-white pebbling price** $BW\text{-Peb}(G)$ of DAG G is minimal cost of any pebbling reaching $(\{z\}, \emptyset)$
- **(Black) pebbling price** $Peb(G)$ is minimal cost of any pebbling reaching $(\{z\}, \emptyset)$ using **black pebbles only** ($W_t = \emptyset$ for all t)

Pebbling Price of Binary Trees

Let T_h denote complete binary tree of height h considered as DAG with edges directed towards root

- Pebbling price of T_h is



$$Peb(T_h) = h + 2$$

(easy induction over the tree height)

- Black-white pebbling price is

$$BW-Peb(T_h) = \left\lfloor \frac{h}{2} \right\rfloor + 3 = \Omega(h)$$

(Lengauer & Tarjan 1980)

Definition of Pebbling Contradiction

CNF formula encoding pebble game on DAG G with sources S , unique target z and all non-source vertices having indegree 2

Associate d variables v_1, \dots, v_d with every vertex $v \in V(G)$

The d th degree pebbling contradiction Peb_G^d over G is the conjunction of the following (written as clauses):

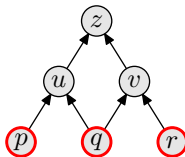
- $\bigvee_{i=1}^d s_i$ for all $s \in S$ (source axioms)
- $(\bigvee_{i=1}^d u_i \wedge \bigvee_{j=1}^d v_j) \rightarrow \bigvee_{l=1}^d w_l$ for all $w \in V(G) \setminus S$, where u, v are the two predecessors of w (pebbling axioms)
- $\bigwedge_{i=1}^d \bar{z}_i$ (target axioms)

Pebbling Contradiction $Peb_{\Pi_2}^2$ for Pyramid of Height 2

$$(p_1 \vee p_2)$$

$$\wedge (q_1 \vee q_2)$$

$$\wedge (r_1 \vee r_2)$$



$$\wedge (\bar{p}_1 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_1 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_2 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_2 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_2 \vee \bar{r}_2 \vee v_1 \vee v_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge \bar{z}_1$$

$$\wedge \bar{z}_2$$

Pebbling Contradiction $Peb_{\Pi_2}^2$ for Pyramid of Height 2

$$(p_1 \vee p_2)$$

$$\wedge (q_1 \vee q_2)$$

$$\wedge (r_1 \vee r_2)$$

$$\wedge (\bar{p}_1 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

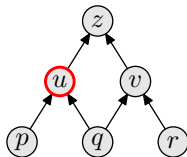
$$\wedge (\bar{p}_1 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_2 \vee v_1 \vee v_2)$$



$$\wedge (\bar{q}_2 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_2 \vee \bar{r}_2 \vee v_1 \vee v_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge \bar{z}_1$$

$$\wedge \bar{z}_2$$

Pebbling Contradiction $Peb_{\Pi_2}^2$ for Pyramid of Height 2

$$(p_1 \vee p_2)$$

$$\wedge (q_1 \vee q_2)$$

$$\wedge (r_1 \vee r_2)$$

$$\wedge (\bar{p}_1 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

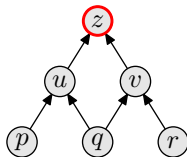
$$\wedge (\bar{p}_1 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_1 \vee u_1 \vee u_2)$$

$$\wedge (\bar{p}_2 \vee \bar{q}_2 \vee u_1 \vee u_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_1 \vee \bar{r}_2 \vee v_1 \vee v_2)$$



$$\wedge (\bar{q}_2 \vee \bar{r}_1 \vee v_1 \vee v_2)$$

$$\wedge (\bar{q}_2 \vee \bar{r}_2 \vee v_1 \vee v_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_1 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_1 \vee z_1 \vee z_2)$$

$$\wedge (\bar{u}_2 \vee \bar{v}_2 \vee z_1 \vee z_2)$$

$$\wedge \bar{z}_1$$

$$\wedge \bar{z}_2$$

Pebbling Contradictions Easy w.r.t. Length and Width

Peb_G^d is an unsatisfiable $(2+d)$ -CNF formula with

- $d \cdot |V(G)|$ variables
- $\mathcal{O}(d^2 \cdot |V(G)|)$ clauses

Can be refuted by deriving $\bigvee_{i=1}^d v_i$ for all $v \in V(G)$ inductively in topological order and resolving with target axioms $\bar{z}_i, i \in [d]$

It follows that

- $L(F \vdash 0) = \mathcal{O}(d^2 \cdot |V(G)|)$
- $W(F \vdash 0) = \mathcal{O}(d)$

(Ben-Sasson et al. 2000)

What about Pebbling Contradictions and Space?

Upper bounds:

- **Arbitrary DAGs G**

optimal black pebbling of G + proof from previous slide:

$$Sp(\text{Peb}_G^d \vdash 0) \leq \text{Peb}(G) + \mathcal{O}(1)$$

- **Binary trees T_h**

improvement by Esteban & Torán (2003):

$$Sp(\text{Peb}_{T_h}^2 \vdash 0) \leq \left\lceil \frac{2h+1}{3} \right\rceil + 3 = \frac{2}{3} \text{Peb}(T_h) + \mathcal{O}(1)$$

- **Only one variable / vertex**

Ben-Sasson (2002):

$$Sp(\text{Peb}_G^1 \vdash 0) = \mathcal{O}(1) \text{ for arbitrary } G$$

No lower bounds on space for $d \geq 2$ previously known

Our Results

Theorem

Let $\text{Peb}_{T_h}^d$ denote the pebbling contradiction of degree $d \geq 2$ defined over the complete binary tree of height h . Then the space of refuting $\text{Peb}_{T_h}^d$ in resolution is $\text{Sp}(\text{Peb}_{T_h}^d \vdash 0) = \Theta(h)$.

Corollary

For all $k \geq 4$, there is a family of k -CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $\text{Sp}(F_n \vdash 0) = \Theta(\log n)$.

Proof Idea

Prove lower bounds on space of $\pi : \text{Peb}_G^d \vdash 0$ by

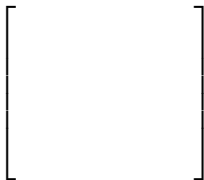
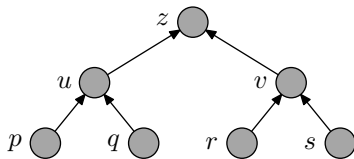
- 1 Interpreting set of clauses $\mathbb{C}_t \in \pi$ in terms of black and white pebbles on G
- 2 Showing that if \mathbb{C}_t induces N black and white pebbles it contains at least N clauses (if $d \geq 2$)
- 3 Establishing that $\pi = \{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ induces black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \dots, \mathbb{P}_\tau\}$ (works only for binary trees T_h)

Then some $\mathbb{C}_t \in \pi$ must induce $BW\text{-Peb}(T_h)$ pebbles

$$\begin{array}{c} \Downarrow \\ |\mathbb{C}_t| \geq BW\text{-Peb}(T_h) = \Omega(h) \\ \Downarrow \\ Sp(\text{Peb}_{T_h}^d \vdash 0) = \Omega(h) \end{array}$$

Developing an Intuition for Black Pebbles

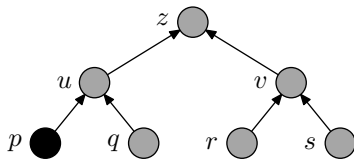
- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |



Empty start configuration

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

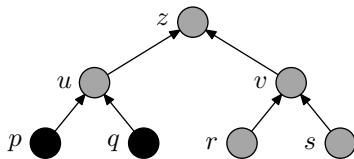


$$\left[\begin{array}{c} p_1 \end{array} \right]$$

Download axiom 1: p_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

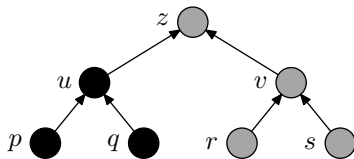


$$\left[\begin{array}{c} p_1 \\ q_1 \end{array} \right]$$

Download axiom 2: q_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

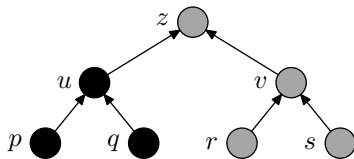


$$\left[\begin{array}{c} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Download axiom 5: $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

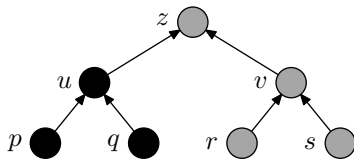


$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee u_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

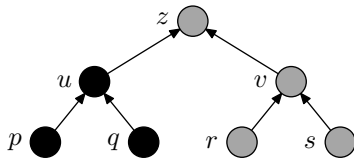


$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee u_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

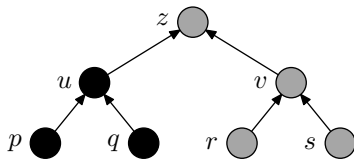


$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

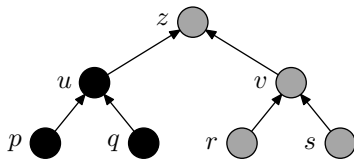


$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

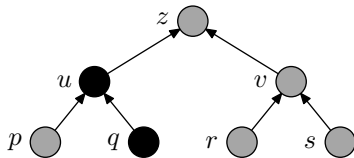


$$\left[\begin{array}{l} p_1 \\ q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause p_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

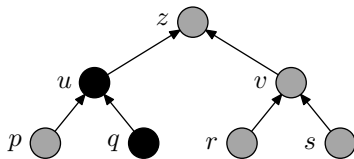


$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Erase clause p_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

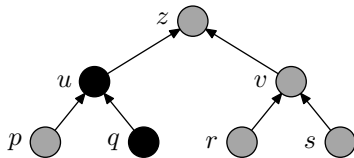


$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer u_1 from
 q_1 and $\bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

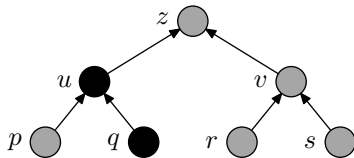


$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \\ u_1 \end{array} \right]$$

Infer u_1 from
 q_1 and $\bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

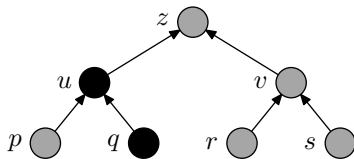


$$\left[\begin{array}{l} q_1 \\ \bar{q}_1 \vee u_1 \\ u_1 \end{array} \right]$$

Erase clause $\bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

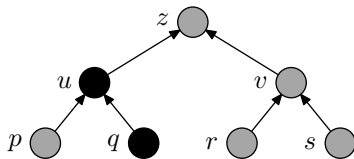


$$\left[\begin{array}{c} q_1 \\ u_1 \end{array} \right]$$

Erase clause $\bar{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

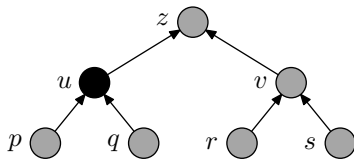


$$\left[\begin{array}{c} q_1 \\ u_1 \end{array} \right]$$

Erase clause q_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

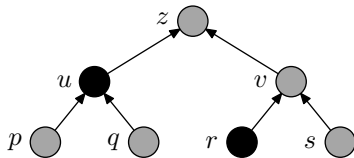


$$\left[\begin{array}{c} u_1 \\ \end{array} \right]$$

Erase clause q_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

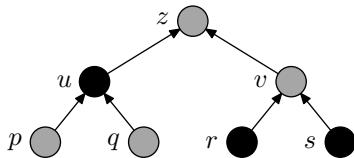


$$\left[\begin{array}{c} u_1 \\ r_1 \end{array} \right]$$

Download axiom 3: r_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

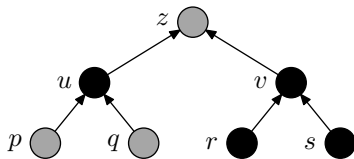


$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \end{array} \right]$$

Download axiom 4: s_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|-----------------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

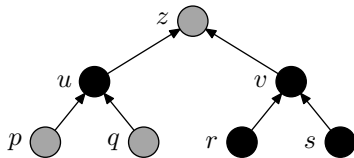


$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Download axiom 6: $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

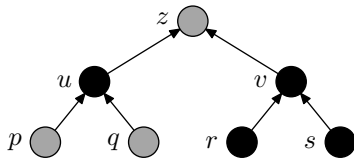


$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee v_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

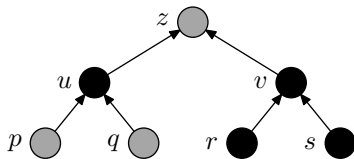


$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee v_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

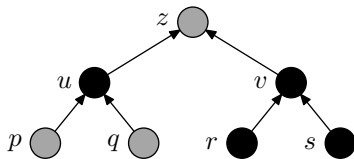


$$\left[\begin{array}{c} u_1 \\ r_1 \\ s_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

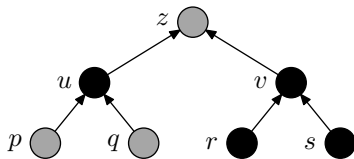


$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

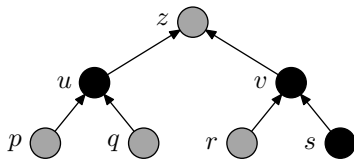


$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause r_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

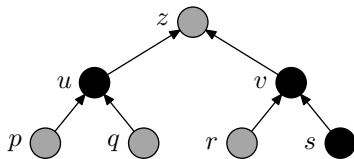


$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Erase clause r_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

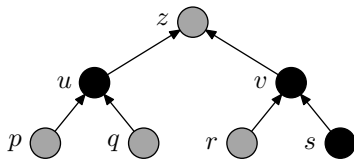


$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer v_1 from
 s_1 and $\bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

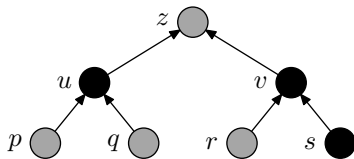


$$\left[\begin{array}{l} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \\ v_1 \end{array} \right]$$

Infer v_1 from
 s_1 and $\bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

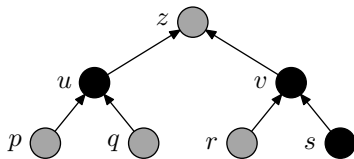


$$\left[\begin{array}{c} u_1 \\ s_1 \\ \bar{s}_1 \vee v_1 \\ v_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

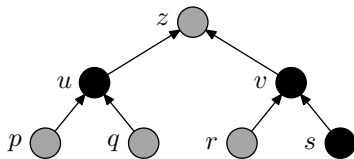


$$\left[\begin{array}{c} u_1 \\ s_1 \\ v_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

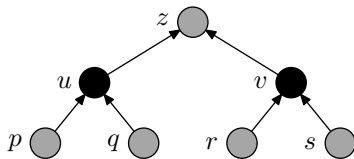


$$\left[\begin{array}{c} u_1 \\ s_1 \\ v_1 \end{array} \right]$$

Erase clause s_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

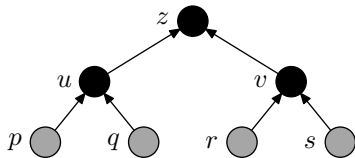


$$\left[\begin{array}{c} u_1 \\ v_1 \end{array} \right]$$

Erase clause s_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|-----------------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

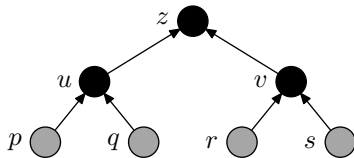


$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \end{array} \right]$$

Download axiom 7: $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

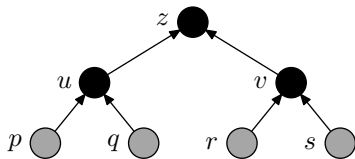


$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{v}_1 \vee z_1$ from
 u_1 and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

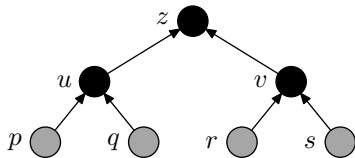


$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{v}_1 \vee z_1$ from
 u_1 and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

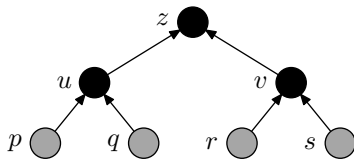


$$\left[\begin{array}{c} u_1 \\ v_1 \\ \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

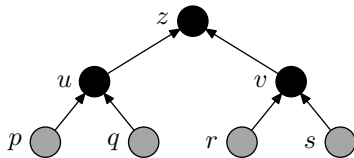


$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

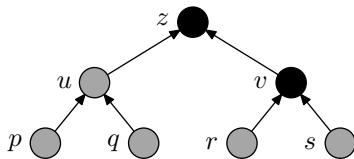


$$\left[\begin{array}{l} u_1 \\ v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause u_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

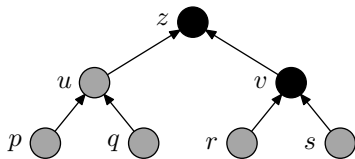


$$\left[\begin{array}{c} v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Erase clause u_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

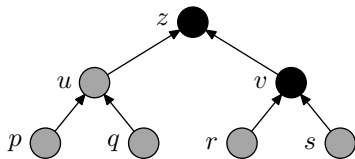


$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \end{array} \right]$$

Infer z_1 from
 v_1 and $\bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

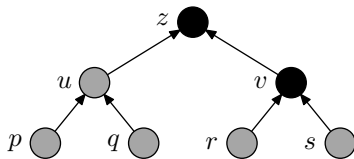


$$\left[\begin{array}{l} v_1 \\ \bar{v}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Infer z_1 from
 v_1 and $\bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

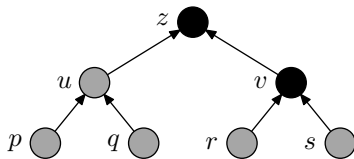


$$\left[\begin{array}{c} v_1 \\ \bar{v}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Erase clause $\bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

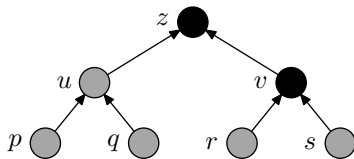


$$\left[\begin{array}{c} v_1 \\ z_1 \end{array} \right]$$

Erase clause $\bar{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

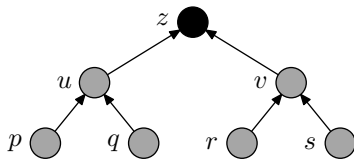


$$\left[\begin{array}{c} v_1 \\ z_1 \end{array} \right]$$

Erase clause v_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

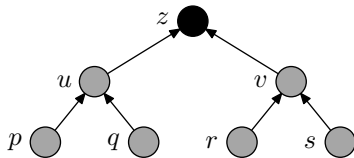


$$\left[\begin{array}{c} z_1 \end{array} \right]$$

Erase clause v_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

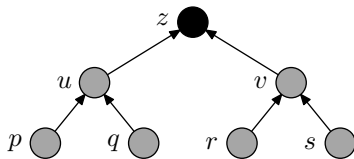


$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Download axiom 8: \bar{z}_1

Developing an Intuition for Black Pebbles

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

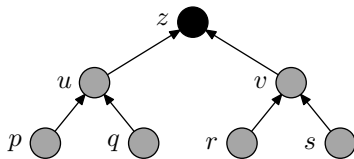


$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Infer 0 from
 z_1 and \bar{z}_1

Developing an Intuition for Black Pebbles

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |



$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \\ 0 \end{array} \right]$$

Infer 0 from
 z_1 and \bar{z}_1

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \models \bigvee_{i=1}^d v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v ?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W , would have single black pebble on v left

Induced White Pebbles

\mathbb{C}_t should induce white pebbles on W below v if **assuming black pebbles on W , we get single black pebble on v**

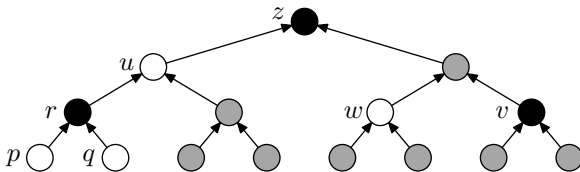
That is, if $\mathbb{C}_t \cup \{ \bigvee_{i=1}^d w_i \mid w \in W \} \models \bigvee_{i=1}^d v_i$.

Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{l|l} \bar{u}_i \vee \bar{w}_j \vee \bigvee_{l=1}^d z_l & 1 \leq i, j \leq d \\ \bar{p}_i \vee \bar{q}_j \vee \bigvee_{l=1}^d r_l & 1 \leq i, j \leq d \\ \bigvee_{l=1}^d v_l & \end{array} \right]$$

to induce the pebbles



Induced Pebbles and Clause Configuration Size

- Formalizing this yields interpretation of clause configuration \mathbb{C}_t derived from Peb_G^d in terms of pebbles on G
- Hope that resolution proof $\pi = \{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ will correspond to black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \dots, \mathbb{P}_\tau\}$ of G under this interpretation
- But to get lower bound on space from this we need to show that

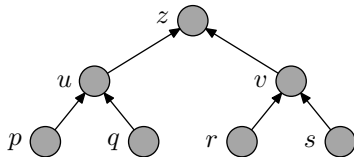
\mathbb{C}_t induces many pebbles



\mathbb{C}_t contains many clauses

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

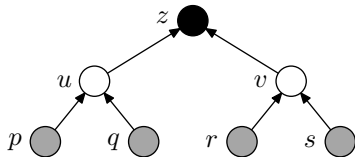


$$\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

Empty start configuration

Not True for $d = 1$ Variable per Vertex

- | | |
|--|-----------------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

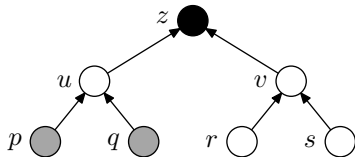


$$\left[\bar{u}_1 \vee \bar{v}_1 \vee z_1 \right]$$

Download axiom 7: $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|-----------------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

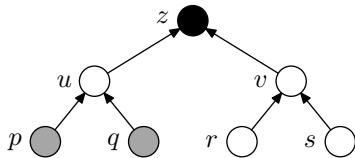


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Download axiom 6: $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

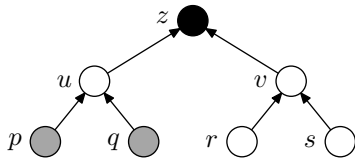


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \end{array} \right]$$

Infer $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$ from
 $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

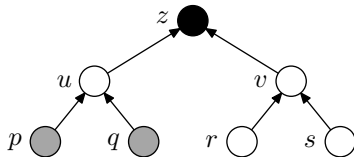


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$ from
 $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ and $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

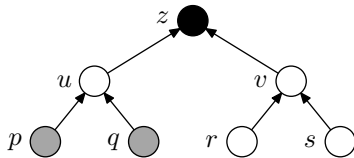


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee v_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

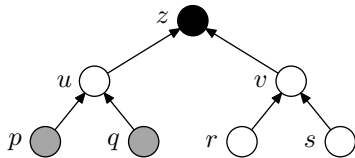


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

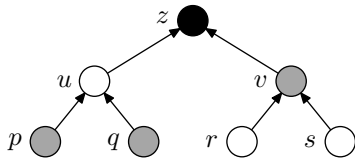


$$\left[\begin{array}{l} \bar{u}_1 \vee \bar{v}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

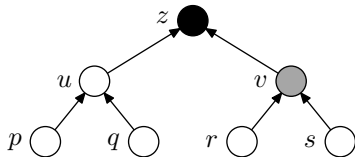


$$\left[\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \right]$$

Erase clause $\bar{u}_1 \vee \bar{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

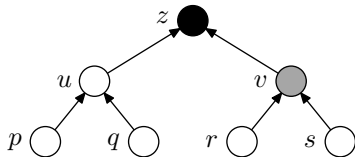


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Download axiom 5: $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

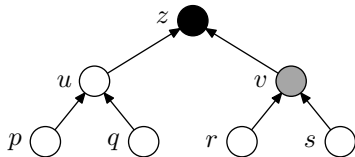


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \end{array} \right]$$

Infer $\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ and $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

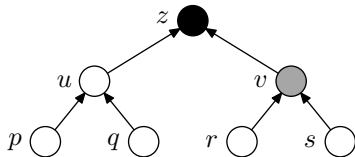


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ and $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

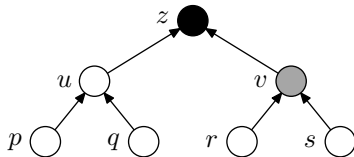


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee u_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

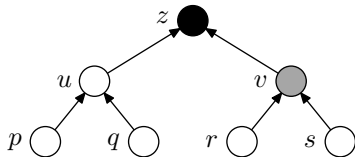


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{p}_1 \vee \bar{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

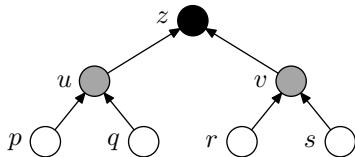


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1 \\ \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

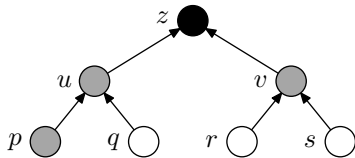


$$\left[\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee \bar{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

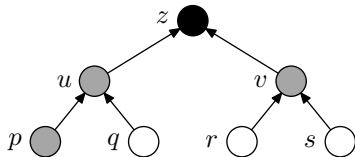


$$\left[\begin{array}{c} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ p_1 \end{array} \right]$$

Download axiom 1: p_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

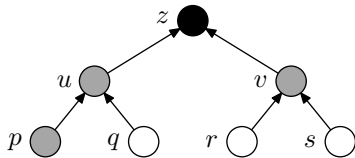


$$\left[\begin{array}{l} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ p_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

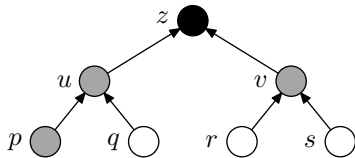


$$\left[\begin{array}{l} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ p_1 \\ \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 p_1 and $\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

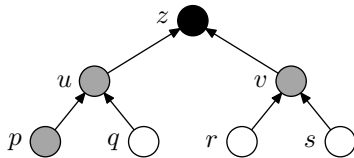


$$\left[\begin{array}{l} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ p_1 \\ \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

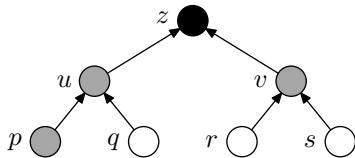


$$\left[\begin{array}{l} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |



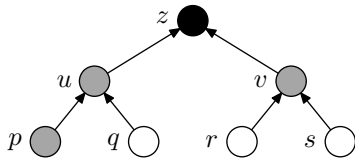
$$\left[\begin{array}{l} \bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause

$$\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |



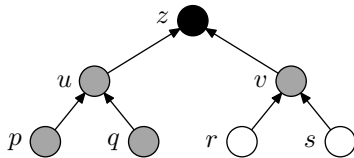
$$\left[\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \right]$$

Erase clause

$$\bar{p}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

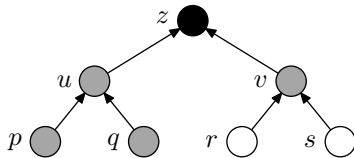


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ q_1 \end{array} \right]$$

Download axiom 2: q_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

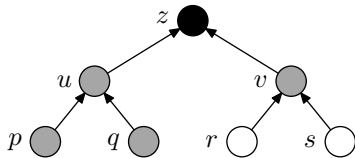


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ q_1 \end{array} \right]$$

Infer $\bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 q_1 and $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

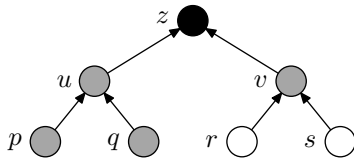


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ q_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{r}_1 \vee \bar{s}_1 \vee z_1$ from
 q_1 and $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

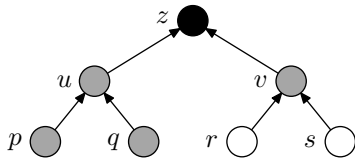


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ q_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

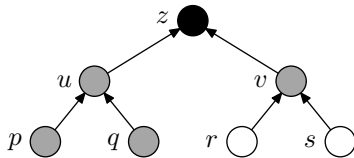


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

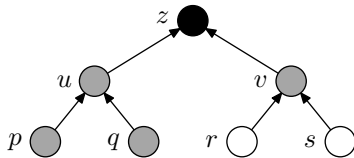


$$\left[\begin{array}{l} \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{r}_1 \vee \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

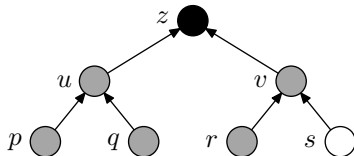


$$\left[\bar{r}_1 \vee \bar{s}_1 \vee z_1 \right]$$

Erase clause $\bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

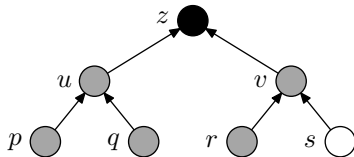


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ r_1 \end{array} \right]$$

Download axiom 3: r_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

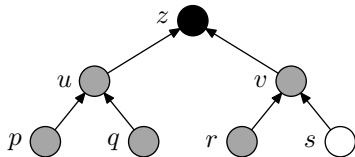


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ r_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee z_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

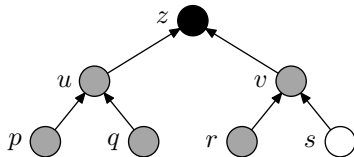


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ r_1 \\ \bar{s}_1 \vee z_1 \end{array} \right]$$

Infer $\bar{s}_1 \vee z_1$ from
 r_1 and $\bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

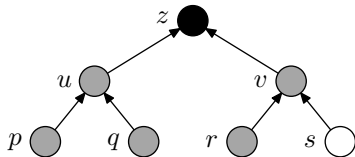


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ r_1 \\ \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

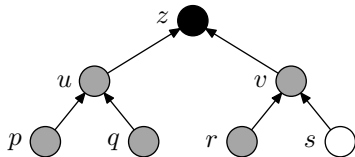


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

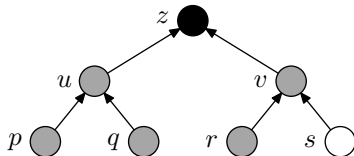


$$\left[\begin{array}{l} \bar{r}_1 \vee \bar{s}_1 \vee z_1 \\ \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

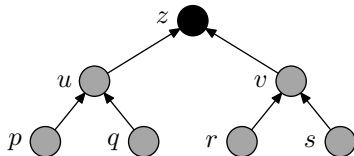


$$\left[\begin{array}{c} \bar{s}_1 \vee z_1 \end{array} \right]$$

Erase clause $\bar{r}_1 \vee \bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | | |
|----|-------------------------------------|----------|
| 1. | p_1 | Source |
| 2. | q_1 | Source |
| 3. | r_1 | Source |
| 4. | s_1 | Source |
| 5. | $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. | $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. | $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. | \bar{z}_1 | Target |

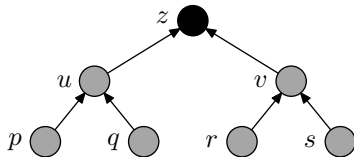


$$\left[\begin{array}{l} \bar{s}_1 \vee z_1 \\ s_1 \end{array} \right]$$

Download axiom 4: s_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

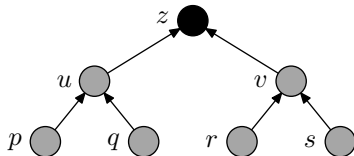


$$\left[\begin{array}{l} \bar{s}_1 \vee z_1 \\ s_1 \end{array} \right]$$

Infer z_1 from
 s_1 and $\bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

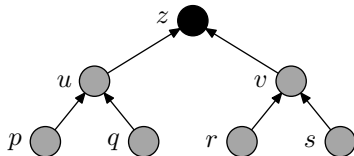


$$\left[\begin{array}{l} \bar{s}_1 \vee z_1 \\ s_1 \\ \mathbf{z_1} \end{array} \right]$$

Infer z_1 from
 s_1 and $\bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

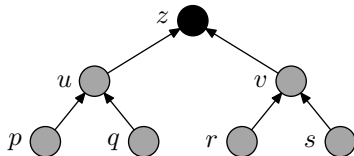


$$\left[\begin{array}{l} \bar{s}_1 \vee z_1 \\ s_1 \\ z_1 \end{array} \right]$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

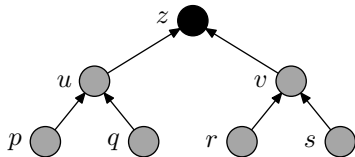


$$\left[\begin{array}{c} \bar{s}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

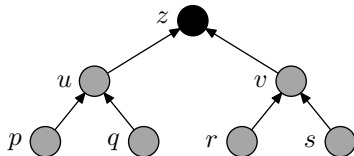


$$\left[\begin{array}{c} \bar{s}_1 \vee z_1 \\ z_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

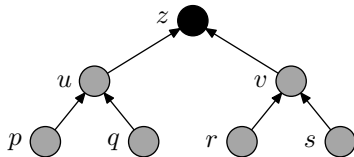


$$\left[\begin{array}{c} z_1 \end{array} \right]$$

Erase clause $\bar{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

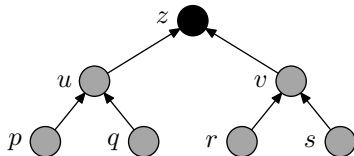


$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Download axiom 8: \bar{z}_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
| 5. $\bar{p}_1 \vee \bar{q}_1 \vee u_1$ | Pebbling |
| 6. $\bar{r}_1 \vee \bar{s}_1 \vee v_1$ | Pebbling |
| 7. $\bar{u}_1 \vee \bar{v}_1 \vee z_1$ | Pebbling |
| 8. \bar{z}_1 | Target |

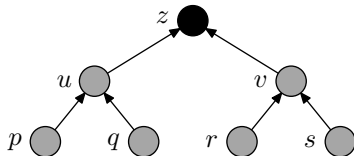


$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \end{array} \right]$$

Infer 0 from
 \bar{z}_1 and z_1

Not True for $d = 1$ Variable per Vertex

- | | |
|--|----------|
| 1. p_1 | Source |
| 2. q_1 | Source |
| 3. r_1 | Source |
| 4. s_1 | Source |
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| 8. \bar{z}_1 | Target |



$$\left[\begin{array}{c} z_1 \\ \bar{z}_1 \\ 0 \end{array} \right]$$

Infer 0 from
 \bar{z}_1 and z_1

But Many Pebbles \Rightarrow Many Clause for $d > 1$

This “top-down” proof in space 3 generalizes to any DAG G

- In terms of our induced pebble configurations:
white pebbles are free for $d = 1$!
- In a sense, this is exactly why $Sp(Peb_G^1 \vdash 0) = \mathcal{O}(1)$
- But for $d > 1$ variables per vertex we can prove that
clauses \geq # induced pebbles

Clauses \geq # Induced Pebbles (Theorem)

F implies D **minimally** if $F \models D$ but $F' \not\models D$ for all $F' \subsetneq F$.

Lemma

Suppose that F implies D minimally. For V any subset of variables, let $F_V = \{C \in F \mid \text{Vars}(C) \cap V \neq \emptyset\}$. Then for all $V \subseteq \text{Vars}(F) \setminus \text{Vars}(D)$ it holds that $|F_V| > |V|$.

Theorem

Suppose that \mathbb{C} is a set of clauses derived from Peb_G^d for $d \geq 2$ and that $V \subseteq V(T)$ is a set of vertices such that \mathbb{C} induces a black or white pebble on each $v \in V$. Then $|\mathbb{C}| \geq |V|$.

Clauses \geq # Induced Pebbles (Informal Argument)

- If \mathbb{C} induces N black pebbles and *no white pebbles* this is *almost* as if \mathbb{C} implied N disjoint clauses
- If so, easy to believe that $|\mathbb{C}| \geq N$
- Problem: When \mathbb{C} induces white pebbles on W , we get a bound not for $|\mathbb{C}|$ but for $|\mathbb{C} \cup \{\bigvee_{i=1}^d w_i \mid w \in W\}|$
- But every white pebble w contributes only 1 clause $\bigvee_{i=1}^d w_i$ but $d > 1$ new variables
- \mathbb{C} must contain $d - 1$ more clauses for every white pebble to eliminate these new variables
- Formalizing this yields the stated lower bound on $|\mathbb{C}|$ in # induced pebbles.

Induced Pebbles Break The Pebbling Rules

Unfortunately, our interpretation of $\pi = \{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ does not yield “well-behaved” pebbling $\mathcal{P} = \{\mathbb{P}_0, \dots, \mathbb{P}_\tau\}$

- Erasures can (and will) lead to large blocks of black and white pebbles suddenly just disappearing
- Need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex

“Illegal” removal of white pebble from w OK if all black pebbles above w dependent on this white pebble are removed as well!

Pebble Subconfiguration $v\langle W \rangle$

Write $v\langle W \rangle$ to denote a **pebble subconfiguration**:

- black pebble on v together with
- white pebbles on W below v *thanks to which we have the black pebble on v*

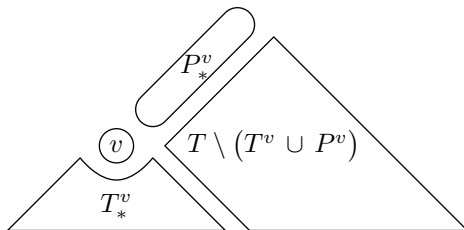
Redefine black-white pebble game in terms of **moves with pebble subconfigurations** instead of individual pebbles

Pebble Subconfiguration Terminology

For a pebble subconfiguration $v\langle W \rangle$:

- White pebbles on $w \in W$ **support** black pebble on v .
- Black pebble on v is **dependent** on white pebbles on W (for $W = \emptyset$ call $v\langle \emptyset \rangle$ an **independent black pebble**).
- A set of pebble subconfigurations \mathbb{L} is a **labelled pebble configuration** or **L-configuration**.

Notation for Vertex Sets in Binary Tree T



$\text{succ}(v)$ immediate successor of v (\emptyset for root z)

$\text{pred}(v)$ immediate predecessors of v (\emptyset for leaf)

T^v vertices in the complete subtree of T rooted at v

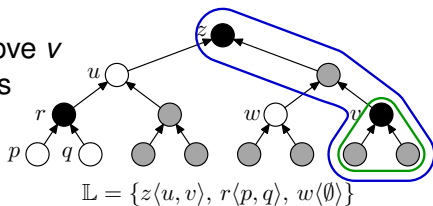
T_*^v T^v without its root, i.e., $T^v \setminus \{v\}$

P^v vertices in the path from v to the root z of T

P_*^v the path without v , i.e., $P^v \setminus \{v\}$

Order Relation on Pebble Subconfigurations

For $u \langle U \rangle$ and $v \langle V \rangle$, if u is above v and U is below V then $u \langle U \rangle$ is **stronger** than $v \langle V \rangle$



Formally:

- $v \langle V \rangle \preceq u \langle U \rangle$ if $T^v \setminus \bigcup_{w \in V} T^w \subseteq T^u \setminus \bigcup_{w \in U} T^w$
- $v \langle V \rangle \prec u \langle U \rangle$ if $v \langle V \rangle \preceq u \langle U \rangle$ and $v \langle V \rangle \neq u \langle U \rangle$

Note that $v \langle \emptyset \rangle \prec z \langle u, w \rangle$ in picture above

Informal Description of Our Modified Pebble Game

- **Pebble placement**: always black on v together with whites on $pred(v)$, except for leaves where $pred(v) = \emptyset$
- **Removal** of white pebbles always allowed, but if so any black pebble dependent on these white pebbles must be removed as well
- “Traditional” removal of white pebble from w corresponds to **merger** of $v\langle V \rangle$ and $w\langle W \rangle$ into $v\langle (V \cup W) \setminus \{w\} \rangle$ followed by **erasure** of $v\langle V \rangle$ and $w\langle W \rangle$
- “Backward” pebbling moves to weaker pebble configurations possible—**reversal** moves

Formal Definition of Our Modified Pebble Game

Definition (Labelled pebble game)

A **labelled black-white pebbling**, or **L-pebbling**, is a sequence $\mathcal{L} = \{\mathbb{L}_0, \dots, \mathbb{L}_\tau\}$ of L-configurations \mathbb{L}_t such that $\mathbb{L}_0 = \{\emptyset\}$ and \mathbb{L}_t is obtained from \mathbb{L}_{t-1} by one of the following:

Introduction $\mathbb{L}_t = \mathbb{L}_{t-1} \cup \{v \langle \text{pred}(v) \rangle\}$ for $v \langle \text{pred}(v) \rangle \notin \mathbb{L}_{t-1}$

Merger $\mathbb{L}_t = \mathbb{L}_{t-1} \cup \{v \langle (V \cup W) \setminus \{w\} \rangle\}$ for
 $v \langle V \rangle, w \langle W \rangle \in \mathbb{L}_{t-1}$ with $w \in V$

Reversal $\mathbb{L}_t = \mathbb{L}_{t-1} \cup \{v \langle V \rangle\}$ provided $v \langle V \rangle \prec u \langle U \rangle$ for
some $u \langle U \rangle \in \mathbb{L}_{t-1}$

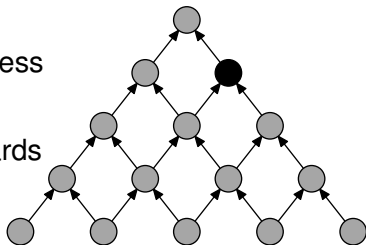
Erasure $\mathbb{L}_t = \mathbb{L}_{t-1} \setminus \{v \langle V \rangle\}$ for $v \langle V \rangle \in \mathbb{L}_{t-1}$

L-pebbling price $L\text{-Peb}(T)$ defined as before

Pebbling Price in Labelled Pebble Game

Reversal moves might seem harmless
but are **dangerous**

- White pebbles may slide upwards and black pebbles may slide downwards
- **Destroys pebbling price for general graphs**
- But still **pebbling price $\Omega(h)$ for binary trees T_h**



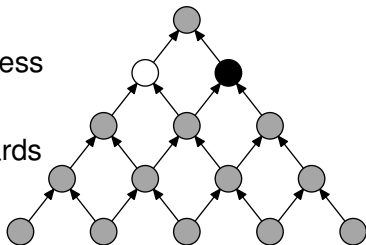
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For T_h a complete binary tree of height h ,
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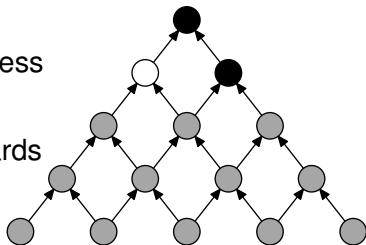
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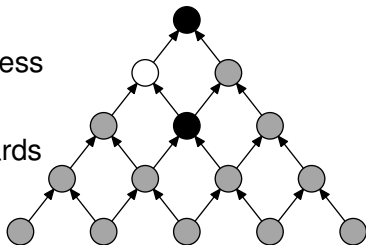
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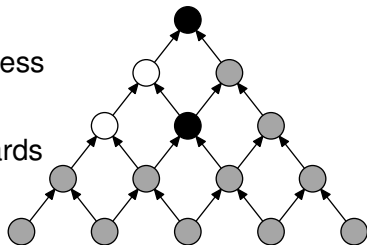
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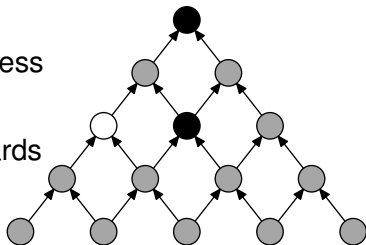
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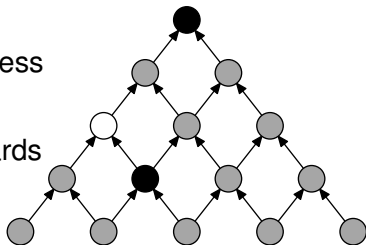
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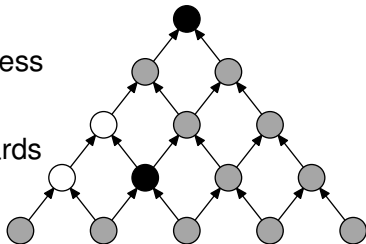
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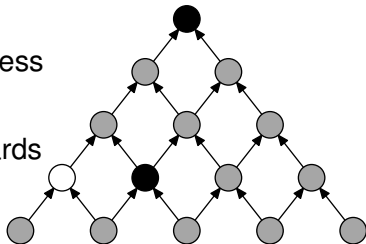
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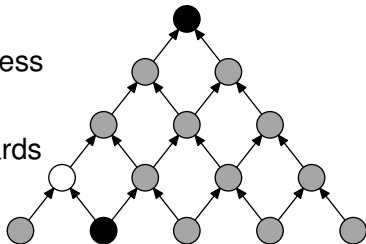
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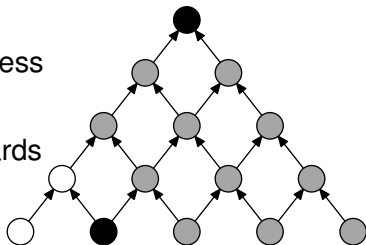
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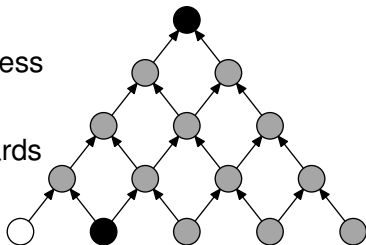
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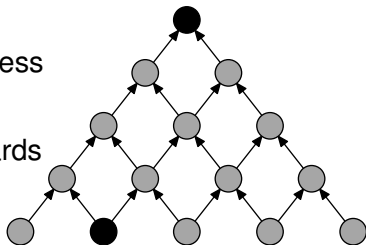
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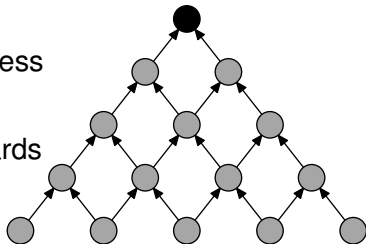
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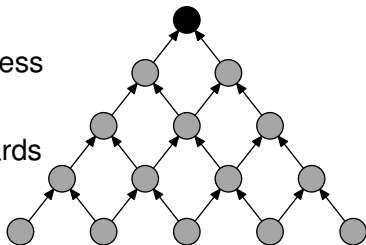
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Theorem

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Formal Definition of Induced Pebble Subconfigurations

Formal definition of induced pebbles turns out quite involved:

Definition (Induced L-configuration)

If for a vertex v there is a *minimal* set $W' \subseteq T \setminus P^v$ such that

- $\mathbb{C} \cup \{ \bigvee_{i=1}^d w_i \mid w \in W' \} \models \bigvee_{u \in P^v} \bigvee_{i=1}^d u_i$ but
- $\mathbb{C} \cup \{ \bigvee_{i=1}^d w_i \mid w \in W' \} \not\models \bigvee_{u \in P_*^v} \bigvee_{i=1}^d u_i$

then \mathbb{C} induces the pebble subconfiguration $v \langle W \rangle$ for

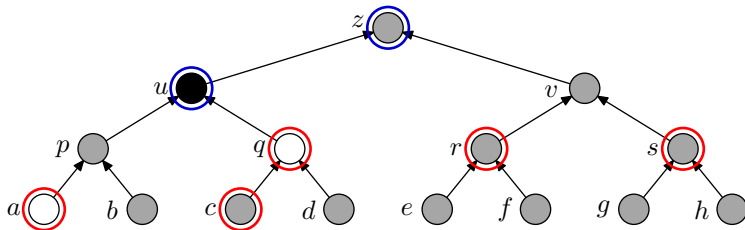
$$W = \{ w \in W' \cap T_*^v \mid P_*^w \cap W' = \emptyset \}.$$

The **induced L-configuration** $\mathbb{L}(\mathbb{C})$ of a clause configuration \mathbb{C} consists of all such induced pebble subconfigurations $v \langle W \rangle$.

Example Induced Pebble Subconfiguration

For $d = 1$, the clause configuration

$$\mathbb{C} = [\bar{a}_1 \vee \bar{c}_1 \vee \bar{q}_1 \vee \bar{r}_1 \vee \bar{s}_1 \vee u_1 \vee z_1]$$

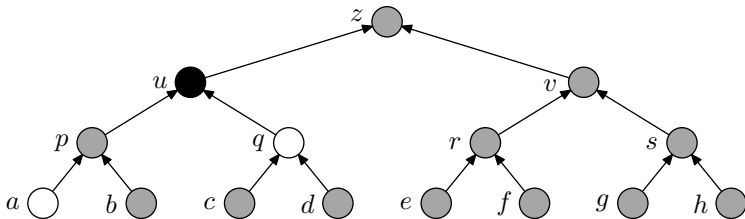


induces the pebble subconfiguration $u \langle a, q \rangle$

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induces the pebble subconfiguration $u \langle a, q \rangle$

Resolution Derivations Induce Labelled Pebblings

Technical detail: study derivations of $\bigvee_{i=1}^d z_i$ from

$*Peb_T^d = Peb_T^d \setminus \{\bar{z}_1, \dots, \bar{z}_d\}$ instead

(Same space $Sp(Peb_{T_h}^d \vdash 0) = Sp(*Peb_{T_h}^d \vdash \bigvee_{i=1}^d z_i)$ anyway)

Theorem

Let $\pi = \{C_0, \dots, C_\tau\}$ be a derivation of $\bigvee_{i=1}^d z_i$ from $*Peb_T^d$.

Then the induced L -configurations $\{L(C_0), \dots, L(C_\tau)\}$ form the “backbone” of an L -pebbling \mathcal{L} of T in the following sense:

All transitions $L(C_t) \rightsquigarrow L(C_{t+1})$ can be done with L -pebbling moves in such a way that $cost(\mathcal{L}) = \mathcal{O}(\max_{t \in [\tau]} \{cost(L(C_t))\})$.

Main Theorem

Theorem

The space of refuting the pebbling contradiction of degree $d \geq 2$ over the complete binary tree of height h in resolution is $Sp(Peb_{T_h}^d \vdash 0) = \Theta(h)$.

Proof sketch.

- Upper bound easy (use “black-pebbling” resolution proof)
- For lower bound, let $\pi = \{\mathbb{C}_0, \dots, \mathbb{C}_\tau\}$ be derivation of $\bigvee_{i=1}^d z_i$ from $*Peb_{T_h}^d$ in minimal space = $Sp(Peb_{T_h}^d \vdash 0)$
- Then there is some $\mathbb{C}_t \in \pi$ that induces $\Omega(h)$ pebbles in T_h
- Thus $Sp(\pi) \geq |\mathbb{C}_t| \geq \#$ pebbles induced by $\mathbb{C}_t = \Omega(h)$. \square

A Separation of Space and Width in Resolution

Corollary

For all $k \geq 4$, there is a family of k -CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof.

We know $W(\text{Peb}_G^d \vdash 0) = \mathcal{O}(d)$ for all G .

Fix $d \geq 2$, let $F_n = \text{Peb}_{T_h}^d$ for $h = \lfloor \log(n+1) \rfloor$ and use the Main Theorem. □

Conclusion

- First lower bound on space in resolution which is not the consequence of a lower bound on width but instead separates the two measures
- Answers an open question in several previous papers
- We believe that it should be possible to strengthen this result in (at least) two ways

Open Problems

Extend to arbitrary DAGs

Conjecture 1

For G an arbitrary DAG with a unique target and with all vertices having indegree 0 or 2, if $d \geq 2$ it holds that $Sp(Peb_G^d \vdash 0) = \Omega(BW-Peb(G))$.

Would yield almost optimal separation $\Omega(n/\log n)$ between space and width

Best conceivable is $\Omega(n)$

Open Problems (cont.)

Generalize to k -DNF resolution and prove space hierarchy

k -DNF resolution: lines in proof not disjunctive clauses but disjunctions of conjunctions of size $\leq k$

Conjecture 2

For k -DNF resolution refutations of pebbling contradictions defined over complete binary trees T_h of height h , fixing k it holds that $Sp_{\text{Res}(k+1)}(\text{Peb}_{T_h}^{k+1} \vdash 0) = \mathcal{O}(1)$ but

$$Sp_{\text{Res}(k)}(\text{Peb}_{T_h}^{k+1} \vdash 0) = \Omega(h).$$

Would show that k -DNF resolution proof systems for increasing k form strict space hierarchy

Related Question for k -DNF Resolution

- For minimally unsatisfiable CNF formulas it is well-known that $\# \text{ clauses} > \# \text{ variables}$
- What about size of **minimally unsatisfiable sets of k -DNF formulas** in terms of number of variables?
- Good lower bound would probably be enough to prove space hierarchy for k -DNF resolution

References

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ECCC Technical Report TR05-066

Available at

`eccc.hpi-web.de/eccc-reports/2005/TR05-066/`

Extended abstract in proceedings of 38th ACM Symposium on
Theory of Computing (STOC '06)

That's basically it!
(modulo all the gory technical details)