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A numerical study of the two-fluid models for dispersed two-phase flow

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Abstract

In this thesis the two-fluid (Eulerian/Eulerian) formulation for dispersed two-phase flow is considered. Closure laws are needed for this type of models. We investigate both empirically based relations, which we refer to as a non-granular model, and relations obtained from kinetic theory of dense gases, which we refer to as a granular model. For the granular model, a granular temperature is introduced, similar to thermodynamic temperature. It is often assumed that the granular energy is in a steady state, such that an algebraic granular model is obtained.

The inviscid non-granular model in one space dimension is known to be conditionally well-posed. On the other hand, the viscous formulation is locally in time well-posed for smooth initial data, but with a medium to high wave number instability. Linearizing the algebraic granular model around constant data gives similar results. In this study we consider a couple of issues.

First, we study the long time behavior of the viscous model in one space dimension, where we rely on numerical experiments, both for the non-granular and the algebraic granular model. We try to regularize the problem by adding second order artificial dissipation to the problem. The simulations suggest that it is not possible to obtain point-wise convergence using this regularization. Introducing a new measure, a concept of 1-D bubbles, gives hope for other convergence than point-wise.

Secondly, we analyse the non-granular formulation in two space dimensions. Similar results concerning well-posedness and instability is obtained as for the non-granular formulation in one space dimension. Investigation of the time scales of the formulation in two space dimension suggests a sever restriction on the time step, such that explicit schemes are impractical.

Finally, our simulation in one space dimension show that peaks or spikes form in finite time and that the solution is highly oscillatory. We introduce a model problem to study the formation and smoothness of these peaks.

Preface

This thesis consists of an introduction and five papers. The five papers are:

Paper I: Reynir L. Gudmundsson and Jacob Yström, *Numerical experiments with two-fluid equations for particle-gas flow.* Paper I in technical report, Department of Numerical Analysis and Computer Science, Royal Institute of Technology (KTH), TRITA-NA-0222, (2002).
Presented by the author at "AMIF 2002, Applied Mathematics for Industrial Flow Problems, Third International Conference, Lisbon, Portugal April 17–20 2002."

The author of this thesis contributed to ideas presented, performed the numerical simulation and wrote parts of the paper.

Paper II: Reynir L. Gudmundsson and Björn Sjögreen. *Numerical experiments with two-fluid equations for particle-gas flow II: Algebraic granular model.* Technical report, Department of Numerical Analysis and Computer Science, Royal Institute of Technology (KTH), TRITA-NA-0446, (2004).

The author of this thesis contributed to ideas presented, performed the numerical simulation and wrote the paper.

Paper III: Reynir L. Gudmundsson. *On the well-posedness of the two-fluid model for dispersed two-phase flow in 2D.* Technical report, Department of Numerical Analysis and Computer Science, Royal Institute of Technology (KTH), TRITA-NA-0223, (2002).

Paper IV: Reynir L. Gudmundsson and Björn Sjögreen. *Viscous formulation of the two-fluid model for dispersed two-phase flow in 2–D.* Technical report, Department of Numerical Analysis and Computer Science, Royal Institute of Technology (KTH), TRITA-NA-0503, (2005).

The author of this thesis had the main responsibility for the mathematical analysis in section 2 and wrote that section of the paper.

Paper V: Björn Sjögreen, Katarina Gustavsson and Reynir L. Gudmundsson. *A model for peak formation in the two-phase equations.* Technical report, Department of Numerical Analysis and Computer Science, Royal Institute of Technology (KTH), TRITA-NA-0433, (2004).
Parts of the paper as been presented at "HYP2004, Tenth International Conference on Hyperbolic Problems Theory, Numerics, Applications, Osaka, Japan September 13-17, 2004"

The author of this thesis contributed to ideas presented and (to a lesser degree) contributed to the writing.

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Contents

Chapter 1

Introduction to two-phase flow

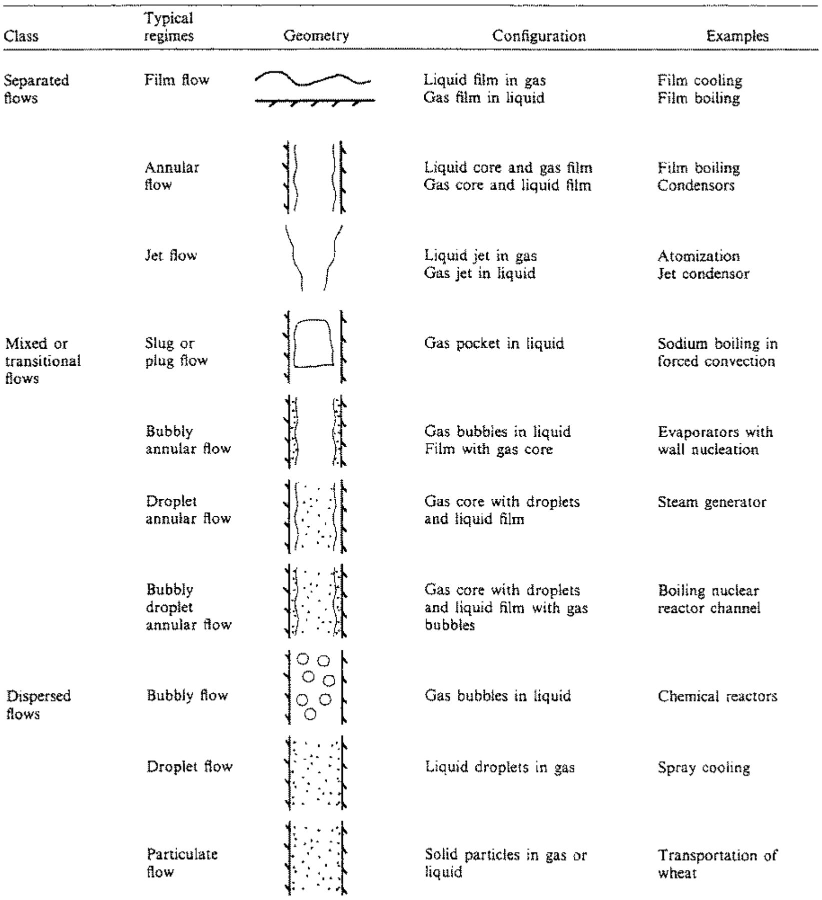
Multiphase flow is a quite common phenomenon, it occurs both in nature and in technology. One of the most trivial example in nature is that clouds are droplets of liquid moving in gas. There are also numerous examples where multiphase flow occurs in industrial applications, for examples energy conversion, paper manufacturing, food manufacturing and medical applications. Due to the large number of applications where multiphase flow occurs, it is important to have accurate models.

Multiphase flow can give rise to very complex combinations of phases as well as flow structures. The simplest case of multiphase flow is *two-phase* flow. Under standard condition there are only four states of matter or phases; that is gas, liquid, solid and plasma phase. In most cases, and in this study, only a mixture of two of the three first phases is considered.

In two-phase flow, complicated interaction between the two phases can occur, for example boiling of water, melting of ice, solidification of metals and many other situations where mass transfer occurs between the phases. The rate of mass transfer can be hard to model. Reaction or mixing of phases can also lead to more complexity of the flow, for example it is very hard to track salt particles in water, since the salt resolves (or diffuses) into the liquid phase. In this study we only consider two-phase flow for immiscible phases and where there is no mass transfer.

There are numerous classification methods in the literature for two-phase flow problems, due to the variety of these. A general classification, by Ishii [25], is to divide two-phase flow into four groups depending on the mixtures of phases in the flow. The four groups are the flow of *gas-liquid*, *gas-solid*, *liquid-solid* and *immiscible liquid-liquid* mixtures. The last case is technically not a two-phase mixture, it is rather a single phase two-component flow, but for all practical purposes it can be considered as a two-phase mixture.

Ishii does also another classification in his work, that is based on the interfacial structure and topological distribution of the flow structure. This classification is more difficult, since the structure of the flow can change continuously. Two-phase flow is here classified according to the geometry of the interface into three classes;



Class	Typical regimes	Geometry	Configuration	Examples
Separated flows	Film flow		Liquid film in gas Gas film in liquid	Film cooling Film boiling
	Annular flow		Liquid core and gas film Gas core and liquid film	Film boiling Condensors
	Jet flow		Liquid jet in gas Gas jet in liquid	Atomization Jet condensor
Mixed or transitional flows	Slug or plug flow		Gas pocket in liquid	Sodium boiling in forced convection
	Bubbly annular flow		Gas bubbles in liquid Film with gas core	Evaporators with wall nucleation
	Droplet annular flow		Gas core with droplets and liquid film	Steam generator
	Bubbly droplet annular flow		Gas core with droplets and liquid film with gas bubbles	Boiling nuclear reactor channel
Dispersed flows	Bubbly flow		Gas bubbles in liquid	Chemical reactors
	Droplet flow		Liquid droplets in gas	Spray cooling
	Particulate flow		Solid particles in gas or liquid	Transportation of wheat

Figure 1.1: Different regimes for two-phase flows according to Ishii [25], taken from [13].

separated flows, transitional or mixed flows and dispersed flows, see Figure 1.1. Note that each of these classes have subclasses defined by typical flow regimes.

This classification is not unique and there are several other classifications for two-phase flows, see for example [24]. The two-phase flow phenomena appears in numerous industrial application so other classifications are often depending on combinations of the two phases as well as on the flow structure. These classifications can be for a specific flow problem, for example in [29] is a classification for fluidization, which is the flow of solid particles in gas or liquid. This is quite common in many industrial applications.

1.1 Models of two-phase flows

When modeling two-phase flow, it is necessary to know what phenomena, effects and flow structures are important. The flow structure is often an important factor. As mentioned above, and illustrated in Figure 1.1, there are three classes of two-phase flow structures. The flow structure differs significantly between these classes, although the mixed or transitional flow is a combination of the other two classes.

Modeling separated and dispersed flows are very different tasks. In a separated flow, the interface between the two phases is quite important and significant. The interface between the phases is of course important also for dispersed flow, but its structure and position is in general hard to track. In some cases the exact structure or position is of no importance, only some average quantity is needed for fluid flow or transport analysis.

Models for two-phase flow can be categorized into two different groups. In the first group, there are models that track the interface between the two phases. These are ideal models for separated flows. In the second group, there are models where the exact position of the interface is not followed specifically. Dispersed flows are usually modeled using models from this second group.

Ideally, one would like to track the interface between the phases at all time, for both the separated and the dispersed flows, similar to resolving all relevant scales of turbulent single phase flow (DNS). This is often too computationally expensive and sometimes redundant. In many cases, especially for dispersed flow, a bulk model is what is sought.

Interface tracking models

For a separated two-phase flow, where the structure of the interface between the two phases is not complexed, it should be possible to use a model for the established single phase flow equations and moving boundary between the phases. However, many flows are too mixed for this method to be usable.

There exist a couple of interface tracking methods that have been applied to two-phase flow problems. In interface tracking methods, the interface is represented by a continuously updated discretization. These discretizations can be of Lagrangian or Eulerian type, depending on the method used.

One early method is the *Marker and Cell* (MAC) method [23]. There, a number of discrete Lagrangian particles are advected by the local flow and the distribution of these particles identifies the regions occupied by a certain phase. There are a couple of other methods based on similar principle, for example a front-tracking method introduced in [47]. That method was designed to simulate the motion of bubbles in a surrounding fluid, but it has been used to simulate other phenomena concerning interface tracking. Those are only some examples of existing methods; a review of similar methods, along with a comparison of them, is presented in [40].

Behind the *level–set* method, introduced in [36], is a different idea compared with particle methods like the MAC method. In the level-set method, which is based on a fixed Eulerian grid, the interfaces are defined as the zero level set of a continuous function that is updated in order to follow the position of the interface.

Each of these methods has its positive sides and its negative sides. The particle methods, like the front tracking method, are easy to implement and the interface force can be applied accurately, but they are usually computationally expensive. On the other hand, using the level–set method it is easy to reconstruct the interface, but this method is not mass conserving in general.

Models for dispersed flows

The complexity of the interfaces between the two phases in dispersed flow is too high for interface tracking methods to be suitable, at least with todays computing capacity. To model this type of flows, another strategy is needed. There are two generic approaches for modeling dispersed flow; the Lagrangian approach and the Eulerian approach.

For particulate (or particle-like) flow, see Figure 1.1, it is possible to construct methods based on an idea similar to the one behind the MAC-method for separated flow. The general idea is to follow each particle of the flow as they advect in the continuous phase. Here, the particles would represent one phase, but not a region as in the MAC-method. This approach is referred to as the *Lagrangian–Eulerian* method, where the continuous phase is calculated in an Eulerian reference frame. Different strategies exist for implementing this idea, see for example, [2], [10] and [37]. Among these there are three different strategies for coupling between the phases, see [9]. In *one-way* coupling, the only influence is on the particle by the surrounding fluid. *Two-way* coupling, the fluid is also influenced by the particle, and in *four-way* coupling, particles are also influenced by each other.

A different way of modeling dispersed flows is to treat both phases as a continuum. This is generally referred to as the *Eulerian–Eulerian* approach or the *two-fluid* model, as discussed in [1],[25]. In this case local instantaneous equations of mass, momentum and energy balance for both phases are derived along with instantaneous jump conditions for interaction between phases. These equations must then be averaged in a suitable way. A volume concentration or volume fraction function is defined. Using this approach introduces more unknowns than equations. Hence, it is necessary to use closure laws. The two-fluid equations are introduced in Section 2.1.

There are two common ways of obtaining these closure laws. The first is obtained by empirical assumptions. The second type is obtained from *kinetic theory of dense gases*. Kinetic theory describing granular matters or granular flow has also been used to model dispersed two-phase flow. Kinetic theory originates from statistical description of ideal and semi-ideal gases, see [8]. Using this theory it is possible to describe the behavior of molecules or particles with well-defined properties and well-defined interactions. This method introduces a granular temperature, which represents the fluctuation or turbulence of the particle velocity. For further informations see for example [32]. A more detailed description of these different type of closure relations are in Section 2.2.

1.2 Outline

This thesis is organized as follows. In Chapter 2 we introduce the governing equations and the necessary closure relations, as mentioned above. The main mathematical properties of these type of models are stated. There are some issues concerning these models, since the inviscid formulation is only conditionally well-posed. Therefore, the two-fluid system is often changed to obtain a well-posed inviscid model, which we will discuss briefly.

In Chapter 3, we make a short overview of the five papers included in this thesis and discuss the purpose and connection between the papers. In Chapter 4, we outline the main conclusions observed in this study. Finally, in Chapter 5, we suggest the natural next steps in the investigation of these two-fluid models.

Chapter 2

Two-fluid model - Eulerian–Eulerian

There are several ways, depending on averaging procedure used, to formulate a two-fluid model. A short description follows on how to do this. This description is an overview and based on [7], [13] and [25].

The general idea to formulate a two-fluid model is to first formulate integral balances for mass, momentum and energy for a fixed control volume containing both phases. These balances must be satisfied at any time and at any point in space, and gives us two types of local equations. The first type are local instantaneous equations for each phase and the second type are expressions for local instantaneous jump conditions at the boundary between the phases. These set of equations could in principle be solved with numerical simulations, that is if the mesh is finer than the smallest length scales and the time step shorter then the time scales of the fastest fluctuations. However, this is not realistic.

Now for example in particle flow, where there are few particles, it would be possible to track the particles in a Lagrangian way and the carrier or continuous phase in an Eulerian way. This is the Lagrangian–Eulerian approach mentioned in Section 1.1. However, this is not always practical, for large number of particles the only practical way is to follow the particle phase in an Eulerian way. Using the Eulerian–Eulerian approach the local instantaneous equations must be averaged in a suitable way, either in space, in time or as an ensemble. These equations can therefore be solved using a coarser mesh and longer time steps, but instead introduces more unknown than the number of equations. Therefore closure laws are needed. A short description on different closure relations are discussed in Section 2.2.

The two-fluid models obtained do not have desired mathematical properties if the usual closure laws are used. Results concerning the two-fluid model studied here are discussed in Section 2.3.

There are certain issues concerning the two-fluid model studied here. Since the inviscid formulation is not unconditionally well-posed for the two-fluid model studied here, other models have been suggested. In Section 2.4 we will review some of these suggested models.

2.1 Governing equations

The three most common averaged procedures that have been used in connection to two-fluid models, are volume, space and ensemble averaging. The ensemble is the most general of these three procedures and the other two can be viewed as approximations of the ensemble averaging. As the names suggest, in the volume average procedure an averaging is done around a fixed point in space at a certain time and similar procedure is for the time averaging. The ensemble average is a little harder to explain, but it can be viewed as the statistical average.

These averaging procedure have their limitations. For example in the case of the volume average, there is a certain restriction on the spatial size where the variables are averaged over. In for example particle gas flow, the characteristic dimension of the averaging volume must be larger than the characteristic dimension of the particle size, but also smaller than the characteristic dimension of the physical systems.

Following these procedures the following averaged equations are obtained for a particle flow (or particle-like flow) in a continuous phase under isothermal conditions. Newtonian properties are often assumed for both phases. Also, assumptions concerning constitutive and transfer laws for this type of two-phase flow are used. The system is written in the following way

$$\begin{aligned}
(\phi^p \rho_p)_t &= -\nabla \cdot (\phi^p \rho_p \mathbf{u}^p), && (2.1)\\
(\phi^c \rho_c)_t &= -\nabla \cdot (\phi^c \rho_c \mathbf{u}^c), && (2.2)\\
\phi^p \rho_p (\mathbf{u}^p_t + \mathbf{u}^p \cdot \nabla \mathbf{u}^p) &= -\phi^p \nabla p + \nabla \cdot \underline{\sigma} - \mathbf{M} + \phi^p \rho_p \mathbf{g}, && (2.3)\\
\phi^c \rho_c (\mathbf{u}^c_t + \mathbf{u}^c \cdot \nabla \mathbf{u}^c) &= -\phi^c \nabla p + \nabla \cdot (\mu_c \phi^c \underline{\dot{\gamma}}^c) + \mathbf{M} + \phi^c \rho_c \mathbf{g}, && (2.4)\\
\phi^p + \phi^c &= 1. && (2.5)
\end{aligned}$$

The system is often closed by

$$\begin{aligned}
\mathbf{M} &= K(\mathbf{u}^p - \mathbf{u}^c), && (2.6)\\
\underline{\sigma} &= -p_p \underline{\mathbf{I}} + \mu_p \phi^p \underline{\dot{\gamma}}^p, && (2.7)
\end{aligned}$$

where

$$\underline{\dot{\gamma}}^j = \frac{1}{2}\left(\nabla \mathbf{u}^j + (\nabla \mathbf{u}^j)^T\right), \quad j = c, p. \qquad (2.8)$$

Here ϕ^p is the volume fraction of a particle (or particle-like) phase, ϕ^c is the volume fraction of a continuous phase, $\mathbf{u}^p$, $\mathbf{u}^c$ the particle and continuous phase velocity vectors respectively and p is the continuous phase pressure. Furthermore ρ_p, ρ_c are the particle and continuous phase densities, μ_p, μ_c are the particle and continuous phase viscosities, $K = K(\phi^p, |\mathbf{u}^p - \mathbf{u}^c|)$ is the drag function between the phases and $\mathbf{g}$ the gravity acting on the phases. Finally, p_p is the particle-collision pressure. Closure laws are needed for μ_p, p_p and K.

This model is generally referred to as Model A. There exist other models, which we will discuss briefly in Section 2.4.

2.2 Closure laws

To close the system (2.1)-(2.8), closure laws are needed for the particle viscosity μ_p, the particle-collision pressure p_p and the drag function K. Here we will only mention the types of closure laws that are used in this thesis [18], [19], [20], [21] and [43]. The closure laws stated here are applicable to many dispersed flow situations, in particular particle-gas flow. First we will describe the empirically based laws, then the so called granular based laws based on kinetic theory.

For clarity, we will refer to the governing equations closed by empirical based closure relation as the *non-granular model* and *granular model* if the closure laws are based on kinetic theory.

Empirically based

There are certain questions how the particle viscosity μ_p should be modeled, but the two main ideas are to assume it constant or as a function of the volume fraction $\mu_p = \mu_p(\phi^p)$, for details see for example [13].

In the literature it is common practice to model the gradient of the particle-collision pressure as a function of the volume fraction and it is often written

$$\nabla(p_p) = G(\phi^p)\nabla\phi^p, \tag{2.9}$$

where G can be thought of as the modulus of elasticity for the particle phase. This function is often modeled empirically, there is a huge difference in magnitude between suggested models in the literature, see for example [35] for comparison. In principle, this function vanish in the dilute limit of small ϕ^p and is practically singular for an upper limit of ϕ^p, usually called the maximum packing ration ϕ_{mp}. Due to this last fact, G is often denoted the hinder function.

In particle-gas flow, the drag force function, that is the momentum transfer between the phases, is often modeled with the following function

$$K = \left(\frac{17.3}{\mathrm{Re}} + 0.336\right)\frac{\rho_c|\mathbf{u}^r|}{d_p}\frac{\phi^p}{(1-\phi^p)^{1.8}}, \tag{2.10}$$

where $\mathbf{u}^r = \mathbf{u}^p - \mathbf{u}^c$ is the relative velocity between phases and the particle Reynolds number, Re, is defined as

$$\mathrm{Re} = \frac{\rho_c(1-\phi^p)|\mathbf{u}^r|d_p}{\mu_c}, \tag{2.11}$$

here d_p is the diameter of the particles. This model is based on the Ergun equation, see [14] and [15].

Granular flow

A common way of closing this system is to use closure laws based on kinetic theory of dense gases, see for example [11], [32] and [48]. There, a granular temperature is introduced, analogous to thermodynamic temperature for gases, to measure the particle velocity fluctuations,

$$\Theta = \frac{1}{3}\langle(\hat{\mathbf{u}}^p)^2\rangle, \tag{2.12}$$

where $\hat{\mathbf{u}}^p$ is the fluctuating particle velocity. A balance of granular energy associated with these particle velocity fluctuations is required to supplement the continuity and momentum balance for both phases. The formulation of the balance equation for the granular energy is,

$$\frac{3}{2}\left[(\phi^p\rho_p\Theta)_t + \nabla\cdot(\phi^p\rho_p\mathbf{u}^p\Theta)\right] = \underline{\sigma} : \nabla\mathbf{u}^p + \nabla\cdot(\kappa\nabla\Theta) - \gamma - J_p, \tag{2.13}$$

where κ the particle thermal conductivity, γ represents the dissipation due to inelastic particle-particle interaction and J_p represents the dissipation or creation of granular energy resulting from the work of the fluctuating force exerted by the gas through the fluctuating velocity of the particles. Finally, $\underline{\sigma}$ is the particle phase stress term given by (2.7).

It is often assumed that the granular energy is in a steady state and dissipates locally, therefore it is possible to neglect convection and diffusion, see for example [4], [46] and [49]. The granular energy balance equation (2.13) is therefore simplified to an algebraic relation for the granular temperature,

$$(-p_p\underline{\mathbf{I}} + \mu_p\phi^p\underline{\dot{\gamma}}^p) : \nabla\mathbf{u}^p - \gamma = 0. \tag{2.14}$$

Here we have inserted the expression for the particle stress, see equation (2.7).

In the literature there is not a general agreement on the form of different terms in the balance equation for the granular energy (2.13). Therefore, we will state some examples how these terms are modeled, see for example [49] for details.

There is a general agreement on how the particle collision pressure p_p is defined and it has the following form

$$p_p = \rho_p\phi^p\left[1 + 2(1+e)\phi^p g_0\right]\Theta, \tag{2.15}$$

where e is the coefficient of restitution and $g_0 = g_0(\phi^p)$ is the radial distribution function. On the other hand, there exists several different models for the shear viscosity, which differ mainly in the dilute limit. Here we show one of them, taken from [32],

$$\begin{aligned}\mu_p = \frac{5\sqrt{\pi\Theta}}{96}\rho_p d_p\Bigg[&\left(\frac{1}{\eta g_0} + \frac{8\phi^p}{5}\right)\left(\frac{1 + \frac{8}{5}\eta(3\eta-2)\phi^p g_0}{2-\eta}\right)\\ &+ \frac{768}{25\pi}\eta(\phi^p)^2 g_0\Bigg],\end{aligned} \tag{2.16}$$

where $\eta = \frac{1}{2}(1+e)$. For the conductivity of the fluctuating energy κ we have the same situation as for the shear viscosity, that is there exists different models, but the biggest deviation is in the dilute limit. Again we show one of these models, taken from [32],

$$\kappa = \frac{25\sqrt{\pi\Theta}}{128}\rho_p d_p \left[\left(\frac{8}{\eta g_0} + \frac{96\phi^p}{5} \right) \left(\frac{1 + \frac{12}{5}\eta^2(4\eta - 3)\phi^p g_0}{41 - 33\eta} \right) + \frac{512}{25\pi}\eta(\phi^p)^2 g_0 \right]. \tag{2.17}$$

The dissipation due to inelastic particle-particle interaction γ is presented in [26] as

$$\gamma = 3(1 - e^2)\rho_p(\phi^p)^2 g_0 \Theta \left[\frac{4}{d_p} \left(\frac{\Theta}{\pi} \right)^{\frac{1}{2}} - \nabla \cdot \mathbf{u}^p \right], \tag{2.18}$$

though the term $\nabla \cdot \mathbf{u}^p$ is typically omitted. Now the dissipation or creation of granular energy J_p is proposed in [31] as

$$J_p = K \left(3\Theta - \frac{K d_p \left(\mathbf{u}^c - \mathbf{u}^p\right)^2}{4\phi^p \rho_p \sqrt{\pi\Theta}} \right), \tag{2.19}$$

where K is the drag force between the phases. Finally, there are also many different models for the radial distribution function g_0, but the expression in [16]

$$g_0 = \frac{3}{5} \left[1 - \left(\frac{\phi^p}{\phi_{mp}} \right)^{\frac{1}{3}} \right]^{-1}, \tag{2.20}$$

where the constant ϕ_{mp} is the maximum volume fraction, coincides with data over the widest range of volume fraction.

The drag force is not modeled with kinetic theory. Here we have to rely on empirically based assumptions and use models based on Ergun equation, see for example equation (2.10).

In very dense flow, sustained contacts between particles occur, which is difficult to model with kinetic theory. The frictional stress must therefore be taken into account in the description of the particle phase stress, see [52]. Frictional stresses are usually assumed Newtonian and are is generally added to the stress predicted by kinetic theory.

2.3 Mathematical characteristics of Model A

The results presented here, concerning the mathematical properties of Model A, do only apply to the non-granular model.

Numerous papers have shown that Model A for the inviscid two-fluid formulation in one space dimension (2.1)-(2.8) is only conditionally well-posed, see for example [12], [16], [33], [34], [45], [51]. That is, depending on the data where the 1–D two-fluid model is linearized around a well-posed or an ill-posed problem is obtained.

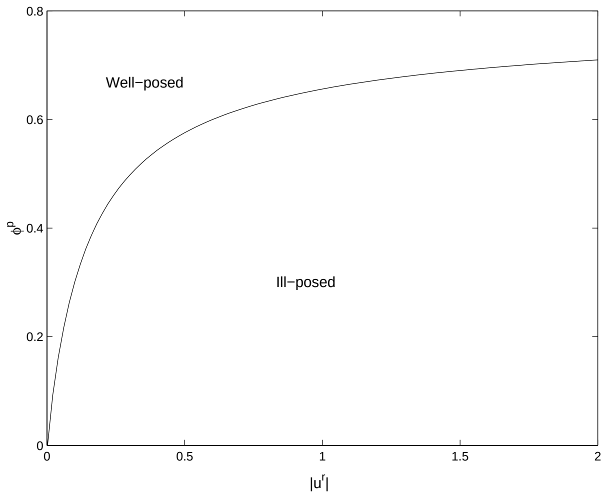


Figure 2.1: Boundary between the region where the inviscid two-fluid model is well-posed and ill-posed, taken from [21].

There is a physically fully reasonable region in phase-space where the inviscid case is ill-posed. The periodic inviscid problem is well-posed if the initial data is smooth and fulfills the hyperbolic condition,

$$G(\phi^p) - (u^r)^2 \frac{\rho_c \rho_p \phi^p}{\rho_c \phi^p + \rho_p (1 - \phi^p)} = \kappa > 0, \tag{2.21}$$

and it is ill-posed if $\kappa < 0$. In Figure 2.1 the boundary between these regions is plotted, that is $\kappa = 0$. Note that in the case of no particle-collision pressure, that is $G(\phi^p) = 0$, the inviscid problem is ill-posed where there is a non-zero relative velocity, $u^r \neq 0$ and both phases are present, $0 < \phi^p < 1$.

Recently, it has been shown that the periodic viscous two-fluid formulation is of mixed hyperbolic-parabolic type and therefore locally in time well-posed if the initial data is smooth, see [51]. The viscous formulation possesses though a medium to high wave number instability.

This means that a smooth solution to the viscous formulation is exponentially unstable. The exponential growth rate of these instabilities are to the first order

$$\alpha = \frac{(u^r)^2 \phi^p (1 - \phi^p) \frac{\rho_c \mu_p^2 (1 - \phi^p) + \rho_p \mu_c^2 \phi^p}{(\mu_c \phi^p + \mu_p (1 - \phi^p))^2} - (1 - \phi^p) G}{\mu_p (1 - \phi^p) + \mu_c \phi^p} \tag{2.22}$$

for $\alpha > 0$. In practice there exists growth where the inviscid case is ill-posed, $\kappa < 0$. Note here that the magnitude of the growth rate is inversely proportional to the

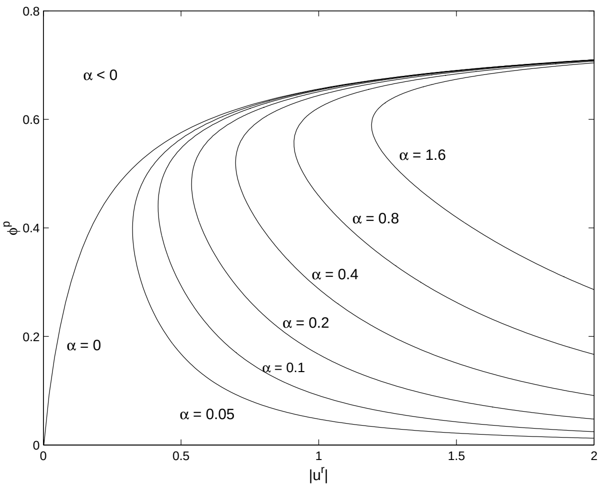


Figure 2.2: Contour plot of the growth rate of the linear instability for particle-gas flow, taken from [21].

viscosities and it increases quadratically with the relative velocity. In Figure 2.2 the growth rate (2.22) is plotted for particle-gas flow. To comparison, the growth rate for particle-liquid flow, where the density and viscosity ratios are significantly different, would be significantly lower.

Computed solutions to the two-fluid model in one, two and three space dimensions do not explode. How can this be possible?

The first thing one must keep in mind interpreting this result is that it is based on linearization around an assumed smooth solution, followed by localization (freezing of coefficients). The result carries over to the non-linear problem as long as the solution is smooth enough. Loosely speaking, a smooth solution in regions of phase space where α is moderate or large is exponentially unstable. It is natural to ask how non-smooth the solution can be so that the principle of linearization followed by localization is still applicable?

Until recently, very little has been investigated about these type of problems. In [28] quasi-linear parabolic problems which are ill-posed in the zero dissipation limit are studied. A simple model problem of this type is considered there

$$u_t + i\sin(nx)u_x = \nu u_{xx}, \quad -\infty < x < \infty, \quad t \geq t_0, \tag{2.23}$$
$$u(x, t_0) = f(x) \tag{2.24}$$

where n is a natural number and $\nu > 0$. Freezing of coefficients, $x = x_0$, gives a growth rate of exponential instability

$$\alpha = \frac{(\sin(nx_0))^2}{4\nu}.$$

However, there is a constant c, such if $|n| \geq \frac{c}{\nu}$, then the solution is uniformly bounded. So, the principle is not valid in this case, see [28] for details.

The are non-linear examples of this type. That is, starting with smooth data, in region of exponential instability, the solution form highly oscillatory structures. These solutions are bounded, see for example [17], [27], [28]

So, there are at least two mechanism that hinder the solution to the two-fluid model (2.1)-(2.8) from exploding like predicted by the analysis of the viscous formulation.

- The solution avoids the region where α is moderate to large.
- The solution becomes highly oscillatory, or strongly dissipative shock-like structures form.

As mentioned above, these results apply only to the non-granular model, but one may ask what can be said about the granular model. In Paper II [19], we study the 1–D algebraic granular model with frictional stresses added, we noted that linearizing the granular model around constant data, yields a model indetical to the linearized non-granular model. Therefore, the results concerning well-posedness and growth holds unchanged for the linearized granular algebraic model.

2.4 Avoiding ill-posedness

Here we will make a brief overview of different two-fluid models that are well-posed in the inviscid limit. That is though often assumed a necessary property for a well-behaved problem, but as stated above the inviscid Model A is only conditionally well-posed. It should be noted that these models are not generally usable for particle gas flow, but there is a similarity between these models and Model A.

There has been some controversy on the form of the pressure term, see for example [6] and [42], which has lead people to present other models, for example Model B. The main difference of Model A and B is that in Model B the continuous phase pressure p is only present in the continuous phase momentum equation (2.4) and not in the particle phase momentum equation (2.3), see for example [16], [33] and [39]. It should be noted that the inviscid Model B is unconditionally well-posed [33], but it is not believed to describe the physics correctly. Model B is sometimes referred to as a *two pressure formulation*, since there are two pressure terms in the two momentum equations, and Model A is often referred to as an *one pressure formulation*.

There exists other types of two pressure formulations, for example the Baer-Nunziato model for deflagration-to-detonation transition, see for example [3]. In the Baer-Nunziato model the two pressures are not required to be in an equilibrium, like in typical two-fluid models. Others have introduced surface tension to model the difference in the two pressure terms and therefore obtain a well-posed inviscid model, see for example [30] and [38].

In [44] an area averaged one-dimensional two-fluid model for pipe or channel flow is investigated. To take into account three-dimensional effects, a momentum flux parameter is introduced and a well-posed inviscid problem is obtained.

Another example is the consolidation of flocculated suspension, which is a gravity dominated flow. Scalar analysis of this problem show that the Froude number is small. The Froude number represents the ratio between the inertia and gravity forces and therefore the convective non-linear terms can be neglected and a well-posed inviscid problem is obtained, see for example [22] and [50] for details.

Chapter 3

Overview of papers

In paper [51] a model problem is studied, that model has the same mathematical properties as the viscous 1-D two-fluid formulation. It is illustrated that steep gradients form in finite time. By regularizing the model problem by adding a second order artificial dissipation to the continuity equation gives a computable solution. It was further observed that the solution depends heavily on the amount of artificial dissipation added and no L_2 convergence seems to hold for long time simulations, as the artificial dissipation is diminished.

Question is; how similar is this model to the full viscous 1-D two-fluid formulation, Model A? Will the full problem behave in a similar way? In paper I [21] and in paper II [19] we make numerical experiments to investigate these questions.

The linear analysis of the 1-D viscous problem predicts exponential growth rate of perturbations of smooth data. Computed solutions in 2-D and 3-D show a highly oscillatory or bubbly behavior, almost of chaotic nature. One may ask; if this behavior is connected by the instabilities seen in the 1-D problem? We think that these things are connected. First step toward understanding the instabilities in 2-D and 3-D is to study the well-posedness of the 2-D model. To our knowledge has this not been done in 2-D for this type of models, which is the reason for the analysis in paper III [18] and in paper IV [20].

Simulations of the two-fluid models have shown formations of peaks or spikes, see for example in our numerical experiments in paper I [21] and in paper II [19]. One might ask if there is a singularity in the solution, but that is the purpose of our investigation in paper V [43].

3.1 Paper I: Numerical experiments with two-fluid equations for particle-gas flow

Here we performed numerical experiments with the viscous incompressible two-fluid formulation, Model A, using closure laws applicable to particle-gas flow. We chose the parameters in these experiments close to physical setup of fluidized bed used in the pharmaceutical industry for coating pellets. Solutions that stay in regions with no or little growth rate are of no interest, there the problem is of standard type. Different type of initial data were considered, starting in regions where exponential growth is expected.

In our simulations we use simple non-dissipative numerical schemes, pseudo-spectral method in space and second order Adam-Bashforth predictor and Adam-Moulton corrector in time. It was observed that steep gradients are formed in finite time. We tried therefore to regularize the problem in a standard way, by adding explicitly a second order artificial dissipation to the continuity equation of the particle phase.

3.2 Paper II: Numerical experiments with two-fluid equations for particle-gas flow II: Algebraic granular model

Here we performed numerical experiments similar to those in paper I [21], except for the granular flow model. The same parameters and numerical scheme are used. We assume that the granular energy is in a steady state and obtain then the algebraic formulation for the granular temperature. Linearizing this model around constant state, one obtains an identical problem as for the non-granular. The problem is a well-posed viscous problem with growth for medium to high frequencies.

Our simulations show that a steep gradients are formed in finite time. As in paper I [21], regularization has not helped, there is no obvious point-wise convergence. However, similar to the non-granular model, the solution seems to stabilize in some sense. Therefore we define new measure, a concept of a 1–D bubble, to obtain an average measure of these seemingly chaotic structures.

3.3 Paper III: On the well-posedness of the two-fluid model for dispersed two-phase flow in 2D

Here we investigate the well-posedness of the two-fluid formulation, Model A, for incompressible inviscid dispersed two-phase flow in two space dimensions. This problem gives a first order closed system of partial differential equations, with one of the equations time independent. These equation are therefore not of standard type. After linearization around constant states and Fourier transformation, one can however eliminate this equation and obtain a closed system of ordinary differential equations for each wave number vector.

3.4 Paper IV: Viscous formulation of the two-fluid model for dispersed two-phase flow in 2–D

This paper studies two problems. First we investigate the well-posedness of the two-fluid formulation, Model A, for incompressible viscous dispersed two-phase flow in two space dimensions. This is done in the same way as the inviscid formulation was studied in paper III [18]. Secondly, we introduce and analyse both an explicit and an implicit numerical scheme. Our analysis of the timescales of the problem indicate that lower order terms impose a sever restriction on the size of the time step, therefore it can be impractical to use an explicit scheme.

We implement the explicit numerical scheme for the two dimensional two-phase equations. the scheme is used to solve simple test problems. The implicit scheme is implemented for the equations in one space-dimension and its performance on simple flow problem is evaluated.

3.5 Paper V: A model for peak formation in the two-phase equations

In this paper we wanted to demonstrate a simple mechanism for creation of oscillations. As observed in paper I [21] and paper II [19] then highly oscillatory structures appear. It might be assumed that these structures are linked to the ill-posedness of the inviscid two-fluid models, but here we consider a model problem that is well-posed in the inviscid limit.

It seems like there is a singularity in the solution to these models and that is an issue that we consider in this paper.

Chapter 4

Conclusions

A couple of interesting things were observed in the numerical experiments of the 1-D formulation performed with the non-granular model in paper I [21] and with the granular model in paper II [19]. First it was found that, for smooth solutions, the lower order terms produce a strong drag force, damping the relative velocity. This rapid damping gives the balance

$$u^r = \frac{\phi^p(1-\phi^p)(\rho_p - \rho_c)g}{K(\phi^p)} \qquad \text{for} \quad u^r \neq 0 \tag{4.1}$$

to high accuracy.

Smooth solutions are not stable, due to the exponential instability. In Figure 4.1 the curve in phase-space defined by (4.1) is plotted, along with the growth rate of the instabilities, this plot is taken from [21]. From simulations we observed that (4.1) was fulfilled almost everywhere, except in intervals where the solution jumped.

The second observation, is that the linear instability generated highly oscillatory structures or patterns, both in time and space. These patterns were not small and after some time spread all over the domain. The number of structures seemed to saturate after some time, and were strongly dependent on the regularization parameter, mentioned in Section 3.1. No point-wise convergence was obtained as the regularization parameter was diminished. These results are in agreement with results obtained for a model problem studied in [51].

The main difference between the results for the non-granular and the granular model was the steepness of the gradient of the volume fraction. The solution to the granular case was somewhat smoother compare to the non-granular case, most likely due to larger particle viscosity.

It was interesting to see that the new averaged quantity, the 1-D mean bubble size, seemed to stabilize along with the saturation of the oscillatory structures. On the other hand, the mean size seemed to depend on the amount of artificial viscosity. It is possible that a convergence in the mean bubble size is obtained for smaller artificial viscosity.

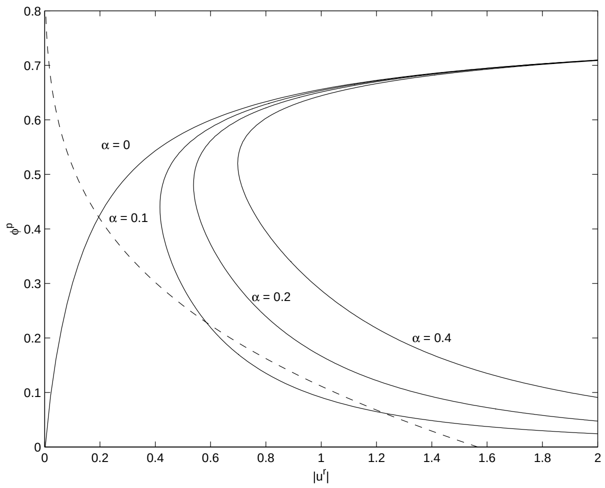


Figure 4.1: Contour plot of the growth rate of the linear instability (*solid*) and line fulfilling Equation (4.1) (*dashed*) for particle-gas flow, taken from [21].

The analysis in paper III [18] and in paper IV [20] gave similar results for the well-posedness of the incompressible model in 2-D compared to the incompressible 1-D results. Both concerning the inviscid and the viscous 1-D results. If the incompressible 2-D model is linearized around constant states Φ, U^p, V^p, U^c, V^c and P, where Φ is the constant state for the volume fraction of the particle phase, ϕ^p. Then in the inviscid limit, the 2-D model is ill-posed if

$$G(\Phi) < \|\mathbf{U}^r\|_2^2 \frac{\rho_c \rho_p \Phi}{\rho_c \Phi + \rho_p (1-\Phi)}, \tag{4.2}$$

where $\mathbf{U}^r = [U^r \quad V^r]^T$ is the relative velocity vector. The viscous 2-D model is locally in time well-posed, but with exponential growth rate of the order

$$\alpha = \frac{\|\mathbf{U}^r\|_2^2 \Phi(1-\Phi) \frac{\rho_p \Phi \mu_c^2 + \rho_c (1-\Phi)\mu_p^2}{(\mu_c \Phi + \mu_p (1-\Phi))^2} - (1-\Phi) G}{\mu_c \Phi + \mu_p (1-\Phi)}. \tag{4.3}$$

This is consistent with the 1-D result, see equation (2.21) and (2.22), which is obtained by substituting

$$\|\mathbf{U}^r\|_2^2 \rightarrow |u^r|^2.$$

The main results from the numerical simulation in paper IV [19] is that the growth rate of viscous problem makes it difficult to solve the problem. This growth rate is due to ill-posedness of the inviscid two-fluid model. It is clear that regularization is needed to obtain a computable solution.

It has also been shown that different time scales effects the problem, that is scales from the viscosity, the convective terms, the particle collision pressure and the source term. To construct a numerical scheme for this problem, it is necessary to know the size of expected scales of the solution.

In paper V [43] it is shown that the solution to the model problem is smooth. On the other hand, the solution has very small scales. It is well possible that same is true for the two-fluid model, since the model problem is derived from it, but as noted in paper I [21] and in paper II [19] we have not been able to estimate the smallest scale.

Chapter 5

Future work

There are a couple of issues concerning the two-fluid model that would be of great interest for further studies in the near future.

From our numerical experiments of both the non-granular and algebraic granular model, we observed that it seems impossible to obtain point-wise convergence for the 1-D formulation, due to the fact that the linearized viscous problem possesses a high wave number instability. The regularization, done by adding second order artificial dissipation, damps the instabilities for the highest wave numbers. Diminishing the artificial dissipation, higher and higher wave numbers are stimulated and the solution becomes increasingly oscillatory. However, the solution seems to stabilize in some sense and stays bounded, even if the number of jumps increases.

One may ask if it is possible to obtain convergence is some other sense, if there is some other quantities or measures where convergence of the regularized problem is obtained. The first observation of our averaged quantity, the 1-D mean bubble size, gives some hope. This quantity seems fairly stable, but we have not yet obtained convergences. Using this concept or testing other ideas should not be that difficult for the 1-D model.

We have seen that the linearized 2-D two-fluid model has similar properties as the 1-D model. For the non-linear problem, these results are only of local type. Therefore, in order to study the long time behavior of these models, numerical experiments will be needed, similar to the work done for the viscous 1-D model in [19] and [21].

In [4] and [5] it is claimed that convergence is obtained for the 2-D two-fluid model by observing that the solution looks similar on two different grids. In those papers, the comparison is done after a relatively short time. Our simulations of the 1-D problem gave a similar convergence for short time studies, but on the other hand no convergence was obtained for the long time behavior. If no point-wise convergence is obtained, a study of bubble statistics could be considered.

Since there was no significant difference between the non-granular model and the algebraic granular model, it would be of interest to consider the full granular model. In [41], a study is done for the 1-D formulation of the full granular model, though for a two pressure formulation of the two-fluid model, which could serve as a starting point.

A study of formation of peaks, or spikes, and oscillatory behavior are of interest. As mentioned, the particle gas flow show bubbly or chaotic behavior. We believe there is a connection between the bubbly behavior and the oscillatory structures. A study of the model problem for peak formation could be of interest for understanding this phenomena.

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