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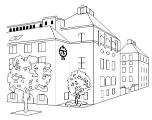
DN2251, Computational Algebra



Reading instructions and Review Questions:

The following instructions and questions are intended to be a help when reading the course and preparing for exam. Among the questions, some may be answered just by reading the text, while some need some hand computation. Even if MATLAB is helpful when preparing the course, only hand calculation is needed for these questions.

Instructions refer to sections in the Demmel text book (D), the Strang text book (S) or my lecture notes (R). The questions are numbered in one sequence.

D 1.5 *Floating point computation:* Backgroud from elementary Numerical Analysis of importance here.

Q 1. Describe a floating point number in terms of sign, fraction, base and exponent.

Q 2. When is a floating point number normalized?

Q 3. What is machine epsilon? What is its approximate value in IEEE double precision?

Q 4. What is the overflow threshold? What is its approximate value in IEEE double precision?

Q 5. What is underflow threshold? What is its approximate value in IEEE double precision?

Q 6. How can one describe the rounding error from one arithmetic operation $+, -, \times, /$ in IEEE arithmetic?

Q 7. How do you bound the rounding error when computing a sum of 3 numbers, $S = a + b + c$?

Q 8. Show that you can bound the forward rounding error when computing a product, $P = a \times b \times c$!

D 1.7 D 2.2 *Vector and matrix norms* are used to bound the perturbation of the solution x caused by a perturbation in the data A and b, when solving the linear system $Ax = b$.

Q 9. Show that $\|x\|_\infty = \max_i |x_i|$ is a vector norm! (Show that it satisfies the 3 conditions for a vector norm!)

Q 10. Show that $\|x\|_p = (\sum |x_i|^p)^{1/p}$, for $p \geq 1$, is a vector norm!

Q 11. Show that for any vector norm $\|x\|$ the expression $\|A\| = \max_{x \neq 0} \|Ax\|/\|x\|$ is a matrix norm, the operator norm!

Q 12. Show that for any operator norm $\|I\| = 1$ where I is the unit matrix!

Q 13. Show that for any operator norm $\|I + hA\| \leq 1 + |h|\|A\|$!

Q 14. Show that for any operator norm $\|(I+hA)^{-1}\| \leq \frac{1}{1-|h|\|A\|}$ if h is small enough! (Hint: Use geometric series expansion $(I-X)(I+X+\cdots+X^{k-1}) = (I - X^k)$!)

Q 15. Let A be a $n \times n$ matrix, $\mathbf{a}$ a column vector with n components and Q an orthogonal $n \times n$ matrix. Show that

$$\|Q\mathbf{a}\|_2 = \|\mathbf{a}\|_2 \qquad \text{and} \qquad \left\|Q^T A Q\right\|_2 = \|A\|_2$$

Q 16. Define condition number of a matrix!

Q 17. Describe what happens with the solution x of the linear system $Ax = b$, when the right hand side b is perturbed into $b + \delta b$!

Q 18. Same as above, if the matrix A is perturbed into $A + \delta A$!

R 1, S 1.2, D 2.3-4 *Linear Systems:* Some of this material has been covered in elementary mathematics courses. The central point is the Gaussian elimination algorithm and its description in matrix terms.

Q 19. Describe Gaussian elimination in three ways:

1. As operations on rows of the matrix.
2. As a pseudocode i. e. in MATLAB style
3. As a premultiplication with elementary elimination matrices

Q 20. What is meant with pivoting for stability?

Q 21. Show that partial (row) pivoting is enough to guarantee that Gaussian elimination can be performed on a square nonsingular matrix!

Q 22. How do you get the factors L and U from the matrix A and the elimination matrices?

Q 23. How many aritmetic operations are needed for computing the Gaussian elimination LU factorization of an $n \times n$ matrix A?

Q 24. How many aritmetic operations are needed to solve a linear system by forward, and back substitution, once the triangular factors L and U are computed?

Q 25. How many arithmetic operations are needed to multiply a $m \times n$ matrix by a vector? Compare with the answer of the previous question!

Q 26. How many arithmetic operations are needed to multiply a $m \times n$ matrix by a $n \times p$ matrix? Compare with the operation count for Gaussian elimination!

D 2.7 S 1.3 *Symmetric positive definite matrices* are common in many important practical applications. Covariance matrices in statistics, normal equations in geodetic surveying problems, as well as stiffness and mass matrices in finite element computation are positive definite.

Q 27. Show that the diagonal elements of a symmetric positive definite matrix are positive!

Q 28. Show that pivoting for stability is not needed for a positive definite matrix! (*Hint:*Look at the answer of the previous question!)

Q 29. For any matrix X the product $A = X^T X$ is positive semidefinite. What is the condition on X that makes A positive definite?

Q 30. For any matrix X the product $C = XX^T$ is also positive semidefinite. What is the condition on X that makes C positive definite?

Q 31. Given a symmetric positive definite $n \times n$ matrix A, and a non-singular $n \times n$ matrix C. Show that $C^T AC$ is symmetric and positive definite.

Q 32. Show that a symmetric positive definite matrix can be factored into LDL^T and that this factorization needs about half the number of flops compared to those needed for a nonsymmetric matrix.

D 2.7.4, S 5.1, R 2 *Sparse matrices* occur as soon as the application involves some type of network structure like in electric grids, transportation networks and finite difference and finite element approximations to partial differential equations.

Q 33. Describe how a graph (V, E) defines a matrix, and how a matrix A gives a graph $G(A)$. Consider only the case of nondirected graphs.

Q 34. How does fill in occur when one does Gaussian elimination on a sparse matrix. Describe it in matrix and graph terms!

Q 35. What is pivoting for sparsity? Why is it done? Describe in matrix and graph terms!

Q 36. Describe the Symmetric Minimum Degree (symmmd) algorithm!

Q 37. Describe the RCM (Reversed Cuthill McKee) algorithm!

D 3.1-2, R 3 *Least squares problems and Singular Value Decomposition*

Q 38. Formulate the least squares problem of fitting a sum of exponential functions $\exp(-\lambda_k t)$, where the decay rates, λ_k, $k = 1, \ldots, p$, are known, to a series of measurements, (y_i, t_i), $i = 1, \ldots, n$. Why do we call this a *linear* least squares problem?

Q 39. Formulate the SVD theorem!

Q 40. What is the range $R(A)$ of a matrix A? How do you find a basis for it by means of SVD?

Q 41. What is the null space $N(A)$ of a matrix A? How do you find a basis for it by means of SVD?

Q 42. What is the condition for full row rank?

Q 43. What is the condition for full column rank?

Q 44. What is the rank of a nonsingular matrix?

Q 45. What is meant by a rank deficient matrix?

Q 46. How can one determine $A^{(k)}$, the matrix of rank k closest to a given matrix A, using the SVD.

Q 47. How can you get the solution x of the least squares problem $\min_x \|Ax-b\|_2$ using the SVD?

Q 48. What are the advantages and disadvantages of replacing the matrix A by a lower rank approximation $A^{(k)}$ when solving a least squares problem?

D 4.2, S 1.5, R 4 *Theory of eigenvalues.* The eigenvalues are roots of a polynomial of degree n, the order of the matrices. This means that there are exactly n complex eigenvalues.

Q 49. A linear system of ordinary differential equations is given by

$$\frac{dx}{dt} = Ax\,, \quad x(0) = x_0$$

Show that its solutions can be expressed in the eigenvalues and eigenvectors of the matrix A! When is the solution stable?

Q 50. What does the position of the eigenvalues of A in the complex plane say about the behaviour of the solutions of the ODE system in the previous question?

Q 51. What is meant by that two matrices are similar?

Q 52. Show that two similar matrices have the same set of eigenvalues. How are the eigenvectors related?

Q 53. Show that if the complex matrix A is Hermitian, the Rayleigh quotient, $\rho(A,x) = \frac{x^H Ax}{x^H x}$ is real for any nonzero complex vector x!

Q 54. Show that all eigenvalues of a Hermitian matrix are real (Hint: use result of previous question!).

Q 55. Formulate the Schur Normal Form.

Q 56. Assume that you have computed the Schur normal form of the matrix A. How can you use it to find the eigenvalues? How do you compute the eigenvectors? When do you not have a full set of n linearly independent eigenvectors?

Q 57. Show that the Schur form of a Hermitian matrix is real and diagonal.

Q 58. What is the value of the Rayleigh quotient $\rho(A,x)$ when x is an eigenvector of A?

D 4.4 S 5.2, R 4 *Transformation algorithms for eigenvalues.* One finite systematic algorithm to make a unitary similarity transformation into Hessenberg form and then one iterative algorithm to get into triangular form.

Q 59. An elementary reflection is defined by $H = I - 2uu^T$. Show that it is orthogonal and symmetric!

Q 60. How do you determine the vector u in the Householder transformation $H = I - 2uu^T$ that transforms a given vector x into another given vector y with the same Euclidean norm $\|y\|_2 = \|x\|_2$?

Q 61. How many arithmetic operations are needed to multiply a full matrix A with a Householder matrix $H = I - 2uu^T$?

Q 62. How do you determine the vector u in the Householder transformation $H = I - 2uu^T$ that transform the matrix A into $A^{(2)} = HAH$ with all but the first two elements in the first column zero?

Q 63. Why does the Householder transformation not go the full way and transform the matrix into triangular form?

Q 64. Why don't you start the QR algorithm right away on the full matrix?

Q 65. You perform the shifted QR algorithm. If you, by chance, have chosen the shift $\tau_k = \lambda_j$ an eigenvalue of A, and compute $A_k - \tau_k I = Q_k R_k, \quad A_{k+1} = R_k Q_k + \tau_k I$, show that the last row of the transformed matrix A_{k+1} will be zero except for its last element that will be λ_j.

Q 66. What is meant with deflation in the QR algorithm?

R 4.2 *Iterative eigenvalue algorithms.*

Q 67. Describe the power method for computing eigenvalues. When does it converge?

Q 68. What is a Krylov sequence?

Q 69. The Arnoldi algorithm has a basic recursion

$$AQ_k = Q_k H_{k,k} + R$$

discuss properties of the basis Q_k, the reduced matrix $H_{k,k}$ and the residual R!

Q 70. Show that if θ is an eigenvalue and s is an eigenvector of $H_{k,k}$, then $y = Q_k s$ gives an approximate eigenvector of A. What is its residual $r = Ay - y\theta$?

Q 71. How can you use the Arnoldi algorithm to solve a linear system of equations $Ax = b$?

Good Luck!