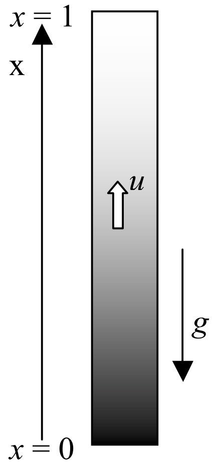
Homework/Lab 3
Due April 17.

A consolidation model



An aqueous suspension of small particles, heavier than water, is left undisturbed under the influence of gravity. The falling particles experience flow resistance, more the higher their concentration.
Let the concentration be $\phi(x,t)$,[-], $0 < \phi < 1$. When $\phi = 1$ the particles form an impermeable continuum. The liquid pressure is p [Pa] and the solids velocity u [m/s]. Then, with g = gravitational acceleration [m/s^2] and $\Delta\rho$ = density difference [kg/m^3] between water and solids,

$$\phi_t + (\phi u)_x = 0 \text{ conservation of mass}$$

$$p_x = (\mu u_x)_x - \Delta\rho\phi g \quad \text{forces: pressure, viscosity, and gravity}$$

$$u = D p_x \text{ pressure gradient proportional to velocity, Darcy's law}$$

Comments:

- You would expect a minus sign in the Darcy law; but there is more than meets the eye here, the model has to take also the fluid velocity into account, and the final result after elimination of the fluid velocity is then … this.
- The model neglects contact forces between particles. This may be reasonable for sand but is certainly not for colloidal suspensions, or surface active particles.

Viscosity μ [Nm/s] and Darcy coefficient D […] depend strongly on concentration. For simplification we will use a constant μ and variable D. Elimination of p leads to the system

$$\begin{aligned} &\phi_t + (\phi u)_x = 0 \text{ hyperbolic} \\ &u = l^2 u_{xx} - \alpha\phi D \text{ elliptic}, l = \sqrt{D(\phi)\mu}, \alpha = \Delta\rho g \end{aligned} \qquad \text{(A)}$$

where the flux in the conservation law is determined from ϕ through the elliptic equation. The boundary conditions are $u(0) = u(1) = 0$. The single characteristic of the hyperbolic equation runs parallel to the boundary, so ϕ should not need boundary conditions. l is small (<< 1) for most materials.

Your task is to write a simulator for the process. The elliptic equation is discretized by second order accurate central differences.
To define a suitable scheme for the hyperbolic equation, consider the limiting case $l = 0$, a scalar non-linear conservation law with flux function f,

$$\phi_t + f_x = 0, f(\phi) = -\alpha\phi^2 D(\phi) \qquad \text{(B)}$$

so we can set $\alpha = 1$ by choosing the time-scale.

A physically reasonable D for (B) should satisfy the conditions

- $D(1) = 0$, impermeable at close packing
- *All* $0 <= \phi < 1$ give finite, positive fall velocity ϕD

1. Taking $D = \phi^{-1}(1-\phi)^b, b > 0$ satisfies some of the above. b = 1 gives

$$f(\phi) = -\alpha\phi(1-\phi), f' = -\alpha(1-2\phi), f'(0) = -\alpha, f'(1) = \alpha$$

so the characteristic changes sign (at ϕ = 1/2). It follows that a symmetric scheme like the Lax-Friedrichs scheme is preferable.

1 a) Implement this for model (A). Take α = 1.

1 b) Find the stability limit for the time step

- for small perturbations around $u = U$, $\phi = \Phi$ (What relation between U and Φ?)
- by numerical experiments

1 c) Take l = 0.1, 0.01, and if you have patience, 0.001. How many cells are necessary to obtain a good solution? Please describe how you determine the "goodness" of a solution. (We shall return to this model later and develop a High-Resolution scheme which will be much less dissipative, so does not need so many cells.)

Exercise 1 c) should convince you of the efficiency of taking l = 0. However, then the solution may develop shocks which we must deal with.

Note: It was an open question for a considerable time if (A) can develop shocks, but it was finally proved by K.Gustavsson and B.Sjogreen in 2003 that it *cannot.*

2. Implement the classical Godunov's method for (B).

2 a) Derive the formulas for an exact Riemann solver. For b = 1 the solution is a shock *or* an expansion wave. Use the properties of f in the manner of L p 228-229 to help develop the solution. The shock is easy: only the sign of the shock speed is needed (Why?). For the wave use the similarity solution $\phi(x,t) = \psi(\xi), \xi = \frac{x}{t}$.

2 b) What values of b comply with the conditions above? Why is b = 1 special and simple? Hint: consider the convexity of f i.e. the monotonicity of f'. Also compare to the shallow water equations of Lab 2.

For (B) the issue of boundary conditions has to be considered again. Why? Hint: Charcteristics. At x = 1 we set ϕ = 0. Try c) and d) at x = 0:

2 c) Set outflow conditions (see L Ch 7). Is this a well posed problem?

2 d) Propose a condition that will allow $\phi(0,t)$ to grow (in its own fashion) from its initial value, approaching 1 only in the long time limit. Hint: The *flux* is zero.

2 e) Check the conservation of ϕ: $\frac{d}{dt}(\int_0^1 \phi(x,t)dx) = 0$. *Propose* a discrete counterpart, *prove* that it is conserved by the Godunov scheme, and *verify* by simulation.