Examination paper Numerical Solution of DE, 2D1255

9-12, May 21, 2007

Read all the questions before starting work. Check carefully that your initially derived equations are correct. Ask if you are uncertain. Answers MUST be motivated. Paginate and write your name on EVERY page handed in.

A total of 17 out of max 34 points guarantees a "pass". The results will be e-mailed to participants by June 1, 2007. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by September 1, 2007, after which the results are irrevocable. The next examination paper will be given in September, 2007.

P1. (7)

Consider a Cauchy-problem for the first order system (*a,b,c* real)

$$\mathbf{q}_t + \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mathbf{q}_x = 0$$

(2) (a) What is the condition on *a,b,c* that the system be hyperbolic? symmetric hyperbolic?

(3) (b) The Lax-Friedrichs scheme for a constant coefficient hyperbolic first order system $\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0$ is

$$\mathbf{Q}_j^{n+1} = \frac{1}{2}\left(\mathbf{Q}_{j+1}^n + \mathbf{Q}_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\mathbf{A}\left(\mathbf{Q}_{j+1}^n - \mathbf{Q}_{j-1}^n\right)$$

Compute the von Neumann magnification matrix **G** for an ansatz

$$\mathbf{Q}_j^n = \hat{\mathbf{Q}}^n(k)e^{ikj\Delta x}$$

What are its eigenvalues? For which values of Δt and Δx is the method stable?

(2) (c) State i) the CFL condition for *non-convergence*, and ii) the Lax Equivalence Theorem, and iii) explain what this has to do with *stability*.

P2. (6)

(3) (a) Derive the upwind method for the advection equation $q_t + uq_x = 0$ from the following algorithm.

1) Reconstruct a piecewise constant function $\tilde{q}^n(x)$ from cell averages Q_j^n

2) Solve $q_t + uq_x = 0, q(x,t^n) = \tilde{q}^n(x), t > t^n$ until $t = t^n + \Delta t = t^{n+1}$

3) Average $q(x,t^{n+1})$ over grid cells to obtain new cell averages Q_j^{n+1}.

(3) (b) Discuss how the reconstruction step can be modified to obtain a high resolution method. You need to explain the use of "slope limiters".

P3. (5)

(2) (a) Linearize the shallow water equations (for horizontal bottom)

$$\begin{cases} h_t + (uh)_x = 0 \\ u_t + uu_x + gh_x = 0 \end{cases}$$

where h = water depth, u = velocity, and g = gravitational acceleration, around $h = H$, $u = U$ (constants). What are the characteristic speeds?

(3) (b) Consider the linearized problem for $0 \le x \le 1$. Suggest boundary conditions at $x = 0$ and $x = 1$ that yield a mathematically well posed problem. If you did not do (a), use $\begin{cases} h_t + ah_x + u_x = 0 \\ u_t + c^2 h_x + au_x = 0 \end{cases}$ Note: There are several cases, dependent on the Froude-number $F = \frac{|U|}{\sqrt{gH}}$ or $\frac{|a|}{|c|}$. Explain!

P4. (8)

(2) (a) Define the meaning of $\mathbf{A}^+$, $\mathbf{A}^-$, and $|\mathbf{A}|$ as used in the Roe-scheme. $\mathbf{A}$ is a real square matrix.
Let the system of conservation laws be $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$.

(1) (b) Define the (…) of a time-stepping scheme in conservation form

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}(...)$$

(2) (c) The numerical flux $\mathbf{F}_{j-1/2}^n$ should satisfy a consistency condition … which?

(1) (d) The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

Define the "flux Jacobian matrix" and its relation to $\mathbf{A}_{j-1/2}$.

(2) (e) Suppose all eigenvalues of $\mathbf{A}$ have equal magnitude, c. Then $|\mathbf{A}|$ becomes very simple. What?

P5. (8)

(2) (a) Define what is meant by *prolongation* and *restriction* in the Multi-grid method.

(3) (b) Give numerical examples for a fine grid with 8 cells and a coarse grid with 4 cells: Write the matrices associated with prolongation by linear interpolation, and the injection and "full weighting" restrictions.

(3) (c) The 1-D elliptic problem $u_{xx} = f(x), u(0) = u(1) = 0$ is discretized by central differences into $u_{j-1} - 2u_j + u_{j+1} = \Delta x^2 f(x_j), j = 1,...,n, u_0 = u_{n+1} = 0$ with solution U_j. The damped Jacobi iteration

$$u_j^{n+1} = (1-\alpha)u_j^n + \alpha(-\Delta x^2 f_j + u_{j-1}^n + u_{j+1}^n)/2$$
$$u_j^0 = 0,$$

is applied. Write the iteration for the error $w_j^n = U_j - u_j^n$. Use von Neumann analysis with ansatz $w_j^n = \hat{w}^n \cdot e^{ikj\Delta x} = \hat{w}^n \cdot e^{ij\theta}, \theta = k\Delta x, 0 < \theta < \pi$ and compute for which α the iteration will converge.