Sample questions.

Four out of five questions on the exam will be very similar to these.

1. Which, (if any) of the following equations are
(a) well posed? (b) hyperbolic? (c) parabolic?

$$u_t + u_x + 2u = 0 \quad (1)$$

$$u_t = u_{xx} - u_{yy} \quad (2)$$

$$\mathbf{q}_t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q}_x = 0, \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix} \quad (3)$$

$$u_t = u + u_{xxx} \quad (4)$$

2. Show that the initial-value problem for $u_t = u_{xx} + u_x + u$ is well posed.
Suppose you want to solve the above equations for $0 \le x \le 1$, $0 \le t \le 1$. State suitable boundary conditions. Show that with your boundary conditions solutions are bounded in norm (which?)

$$\|u(.,t)\| \le C\|u(.,0)\|$$

Here C must be independent of the initial condition $u(x,0)$.

3. Consider a general initial value problem for systems of linear PDE with constant coefficients, in one space dimension, $\mathbf{q}_t = P(\partial_x)\mathbf{q}$. State precise conditions for well-posedness. Give examples of an ill posed and a wellposed problem. Show that the conditions *cannot be* met and *are* met, respectively.

4. Derive the Rankine-Hugoniot condition for a system of conservation laws

$$\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$$

(b) Show that the shock speeds approach the characteristic speeds when the magnitude of the shock jump vanishes.

5. Consider $q_t + \left(q^2/2\right)_x = 0$ together with initial condition
$q(x, 0) = 1$ if $x < 0$, $= a$ if $x > 0$. Solve the initial-value problem for the two cases $a = 2$ and $a = 0$.

6. Consider the system

$$\mathbf{q}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{q}_x = 0, \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$

in $0 \le x \le 1$. Which of the following boundary conditions yield a well posed problem? Discuss reflections in wellposed cases.

(a) $u(0,t) = 1$, $v(0,t) = 0$,
(b) $u(0,t) + v(0,t) = 0$, $u(1,t) = 1$,
(c) $u(0,t) - v(0,t) = 0$, $u(1,t) + v(1,t) = 1$.

7. Consider a scalar conservation law $q_t + (f(q))_x = 0$ approximated by a finite volume method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

where F is called the numerical flux. Consider the following (or the Lax-Friedrichs, or the Lax-Wendroff, or the …) scheme

$$Q_i^{n+1} = \frac{1}{4}\left(Q_{i-1}^n + 2Q_i^n + Q_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(Q_{i+1}^n) - f(Q_{i-1}^n)\right)$$

(a) What is the numerical flux?

(b) What is the order of accuracy?

(c) Consider the scheme applied to the advection equation $q_t + uq_x = 0$ with constant u. For which Δt, Δx is the method stable?

(d) Derive its modified differential equation. Of what type (hyperbolic, parabolic, …) is it? How is the stability limit related to the well-posedness of the modified equation?

8. Consider a scalar advection equation $q_t + \left(u(x)q\right)_x = 0$ approximated by a finite volume method $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$ where F is called the numerical flux. What is F for

(a) the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u_{i-1/2}^+ \Delta Q_{i-1/2}^n + u_{i+1/2}^- \Delta Q_{i+1/2}^n\right),$$

where $u^+ = \max(u,0)$, $u^- = \min(u,0)$, $\Delta Q_{i-1/2} = Q_i - Q_{i-1}$. You need to define $u_{i-1/2}$ etc. properly. Here's a suggestion:

$$u_{i-1/2} = \begin{cases} (u_i Q_i - u_{i-1}Q_{i-1})/\Delta Q_{i-1/2}, & \text{when } \left|\Delta Q_{i-1/2}\right| > 0 \\ (u_i + u_{i-1})/2, & \text{when } \left|\Delta Q_{i-1/2}\right| = 0 \end{cases}$$

(b) the Lax-Friedrichs method

$$Q_i^{n+1} = \frac{1}{2}\left(Q_{i-1}^n + Q_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(u_{i+1}Q_{i+1}^n - u_{i-1}Q_{i-1}^n\right)$$

(c) the Lax-Wendroff method (for constant $u(x) = u$)

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

(d) Describe how these fluxes are used to construct a high resolution method.

9. Derive the upwind method for the advection equation $q_t + uq_x = 0$ from the following algorithm.

1) Reconstruct a piecewise constant function $\bar{q}_n(x)$ from the cell averages Q_i^n.
2) Solve

$$q_t + uq_x = 0, q(x,t_n) = \bar{q}_n(x), t_n < t,$$

until $t = t_n + \Delta t = t_{n+1}$
3) Average over grid cells to obtain new cell averages.
Discuss how this approach can be modified to obtain a high resolution method.

10. Consider $q_t + uq_x = 0$ discretized. A general one-step method can be written as

$$\mathbf{Q}^{n+1} = N(\mathbf{Q}^n), \mathbf{Q} = (\ldots, Q_1, Q_2, \ldots)$$

(a) What does "N is contractive in the 2-norm" mean?

(b) Define the local truncation order in terms of N.

(c) Show that if a method is contractive in a norm then one can obtain an error bound, valid for all $0 \le t_n \le T$, in terms of $\max_{0 \le n \le M} \left\| \tau^n \right\|$ Here, $T = M\Delta t$, where Δt is the timestep and τ^n is the local truncation error.

(d) State in a precise way what is meant by convergence. Explain when convergence follows from your result.

(e) Consider the Lax-Wendroff (or Lax-Friedrichs or Upwind) method. Show that the corresponding N is a contraction in $\left\| \cdot \right\|_2$ if the CFL condition is satisfied.

(f) Consider $q_t + uq_x = q$ Modify the requirements on N and derive a corresponding error bound and convergence result.

11. Consider $q_t + \left(q^2/2\right)_x = 0$ discretized by $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} Q_i^n \left(Q_{i+1}^n - Q_{i-1}^n\right)$

(a) Is this a conservative discretization? If not, suggest a conservative discretization.

(b) Is this a consistent discretization? If so, what is its order of accuracy?

12. Linearize

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + K\rho^\gamma \end{pmatrix}_x = 0$$

at constants $\rho 0$ and $u0$. Here $K > 0$ and $\gamma \ge 1$ are constants.

(a) Derive the Lax-Wendroff method for the linear(ized) system. For which values of Δt and Δx is the method stable? State in a precise way what is meant by stability.

(b) Consider the linearized problem for $0 \le x \le 1$. Suggest boundary conditions at $x = 0$ and $x = 1$ that yield a mathematically well posed problem.

(c) Assume the linearized equations are discretized by the Lax-Wendroff method. Suggest numerical boundary conditions where they are needed.

13. Linearize

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0$$

at constants $h0$ and $u0$. g is the gravitational acceleration.

(a) Derive the Lax-Wendroff method for the linear(ized) system. For which values of Δt and Δx is the method stable? State in a precise way what is meant by stability.

(b) Consider the linearized problem for $0 \le x \le 1$. Suggest boundary conditions at $x = 0$ and $x = 1$ that yield a mathematically well posed problem.

(c) Assume the linearized equations are discretized by the Lax-Wendroff method. Suggest numerical boundary conditions where they are needed.

13. Discretize the initial vaue problem for the real linear hyperbolic system

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0,\ t \ge 0,\ \mathbf{q}(x,0) = f(x)$$

with forward difference in time and central (or forward or backward) difference in space. Analyze the stability using von Neumann analysis.

14. Consider the heat equation $q_t = kq_{xx}$ with $k > 0$. Analyze the stability of

$$a)\quad Q_i^{n+1} = Q_i^n + \alpha\left(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n\right)$$

$$b)\quad Q_i^{n+1} = Q_i^n + \alpha\left(Q_{i-1}^{n+1} - 2Q_i^{n+1} + Q_{i+1}^{n+1}\right)$$ (Hint: Von Neumann)

$$\alpha = \frac{k\Delta t}{\Delta x^2}$$

15. Consider solving the heat equation on a d–dimensional unit box in space with Dirichlet boundary conditions, using second order finite difference approximations for the spatial derivatives. Assume $\Delta x = 1/N$ is the space step in all space directions, and that accuracy in time requires a time step $\Delta t \le \Delta x$. Consider $d = 1$, 2 and 3 and discuss how the work (flops) to compute until $t = 1$ increases with N in the following two cases.
(a) An explicit method is used in time. Remember that stability requirements must be satisfied.
(b) An implicit method (with good stability properties) is used, and in each time level the system of equations is solved using Gaussian elimination.

16. Explain why explicit time stepping usually is used for hyperbolic problems while implicit time stepping is used for parabolic problems.

17. Consider
(1) $u_{tt} + u_{xx} = 0$,
(2) $u_{tt} - u_{xx} = 0$,
(a) Introduce $\mathbf{w} = (u_x, u_t)^T$ and derive systems of first order equations for $\mathbf{w}$ for (1) and (2). Discuss hyperbolicity of the first order systems.
(b) Discuss the possibility of solving (1) and (2) by time-stepping (t is the time-variable) the corresponding first order systems dicretized by the Lax-Friedrichs method.

18. Consider the solution of the ”Poisson equation” in 1D,

$$u_{xx} = f(x), u(0) = 0, u(1) = 0 \qquad (1)$$

discretized by central differences on an equidistant grid, by the multigrid method.
(a) Write the difference equations for the simplest explicit time-stepping scheme with time-step Δt for the initial-value problem

$$u_t - u_{xx} = -f(x), u(0,t) = 0, u(1,t) = 0, u(x,0) = 0 \quad (2)$$

(b) Write the recursion for the classical Jacobi iteration for (1) and show its connection to the time-stepping scheme for (2)
(c) Let the error be $v(x,t) = u(x,\infty) - u(x,t)$. Write the difference equations for the v^n_m. Use Fourier analysis with ansatz function $v_m^n = V_n e^{ikx_m} = V_n e^{i\theta m}, \theta = k\Delta x$. and write the recursion for $V_n(\theta)$.
(d) What range for θ must we consider? What are the high-frequency and the low-frequency ranges? What is the stability limit on Δt?
(e) Suppose the initial data v^0_m is superposed from only high-frequency components. What choice of Δt will most rapidly decrease v^n_m ?

19. Explain the prolongation **P** and restriction **R** operations in the transfer of solutions and residuals between grid levels. Write the *Injection* and *Full weighting* restriction operators for grids with 8 viz. 4 cells. What does the "Galerkin condition" mean?

20. Write the difference stencil for the central difference approximation to the Laplace operator in 2D.

(b) The Laplace equation says

$$\Delta u = 0 \text{ in } \Omega \text{ which is equivalent to } \int_\Gamma \nabla u \cdot n ds = 0 \text{ for}$$

$$\text{any non-selfintersecting, smooth, closed curve } \Gamma \text{ in the domain } \Omega$$

Write the formula for the difference stencil for a finite volume to demonstrate the corresponding discrete conservation property. Draw the gridpoints and the cells.

21. Show that the initial-boundary value strip problem

$$u_t = \alpha u_{xx} + bu, u(x,0) = f(x)$$
$$u(0,t) = 0$$
$$u(1,t) + hu_x(1,t) = 0$$

is well-posed for $h > 0$. Hint: Energy method, consider $\int_0^1 uu_t dx$.

22. The Roe first order method for the system of conservation laws $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$ may be written

$$\mathbf{Q}_m^{n+1} = \mathbf{Q}_m^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}_{m-1/2}^+(\mathbf{Q}_m^n - \mathbf{Q}_{m-1}^n) + \mathbf{A}_{m+1/2}^-(\mathbf{Q}_{m+1}^n - \mathbf{Q}_m^n)\right) \quad (1)$$

(a) Define the meaning of $\mathbf{A}^+$, $|\mathbf{A}|$ and $\mathbf{A}^-$ for a matrix **A**.

(b) Define what is meant by the "*numerical flux* $\mathbf{F}^n_{m-1/2}$" and a "*conservative scheme*"

(c) The Roe-matrix $\mathbf{A}_{m+1/2}$ satisfies

$$\mathbf{A}_{m+1/2}\left(\mathbf{Q}_{m+1} - \mathbf{Q}_m\right) = \mathbf{f}(\mathbf{Q}_{m+1}) - \mathbf{f}(\mathbf{Q}_m)$$

Show that this makes (1) conservative with numerical flux

$$\mathbf{F}_{m-1/2}^n = \frac{1}{2}\left(\mathbf{f}(Q_{m+1/2}^n) + \mathbf{f}(Q_{m-1/2}^n)\right) - \frac{1}{2}\left|\mathbf{A}_{m-1/2}^n\right|\Delta\mathbf{Q}_{m-1/2}$$

23. Define the "Total variation" TV(f) of a function f on $[a,b]$
The sign-function sgn(x) = +1, $x > 0$, and –1, $x < 0$.

(a) What is TV(sgn(sin(x))) over $(-\pi/2, 7\pi/2)$?

(b) Show that d/dt(TV($q(.,t)$)) = 0 on $(-\infty,+\infty)$ when q satisfies the advection equation $q_t + u(x)q_x = 0$, with initial data non-zero only in $[-L,L]$ for some L.

24. Consider the Cauchy-problem

$$q_t + (-xq)_x = 0,$$

with square pulse initial data

$$q(x,0) = 1, |x| < 1, = 0, \text{elsewhere}$$

(a) Solve the equation. **Hint:** Write in the quasi-linear form, use the method of characteristics, and note: the characteristics are NOT straight lines, NOR is the solution constant along them, but the solution is easy anyway.

(b) Compute the total variation TV($q(.,t)$) (can be done without (a))