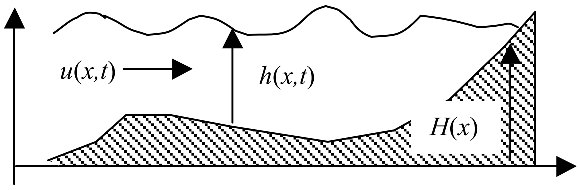
Homework/Lab 4&5, part 1
Deadline May 10 for both Part 1 and Part 2

A Tsunami model - the shallow water equations.

The shallow water model, L. p xxx

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{pmatrix}_x = 0$$

describes water flowing on a horizontal bottom, h = water depth [m], $u = x$-velocity [m/s], g = gravitational acceleration [m/s^2]. Suppose now that the bottom has a shape $H(x)$



An approximate model for this is (not in conservation form)

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + hu_x \\ huu_x + gh(h_x + H_x) \end{pmatrix} = 0$$

- Check that a steady solution (u = 0) must have, as it should, horizontal water level.
- Write the equation in conservation form for h and $m = hu$:

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h,m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h,m,x) \end{pmatrix}$$

It is important that the source function should not contain derivatives of h or m.

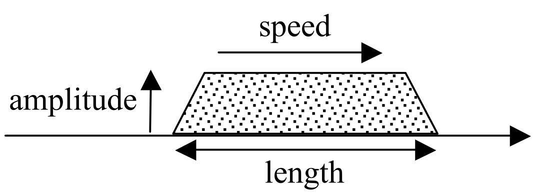
Your job in the first part of the lab. is to write a Roe-solver for the model and simulate a Tsunami. In the second part you shall extend the solver to a high-resolution scheme.

Part 1

We begin with the Tsunami mechanism, as far as it is contained in the (quite crude) model. The Tsunami is generated by a sudden vertical change (say ΔH = 1 m over a length of d = 1000 m) of the bottom topography in a deep ocean (h_0 = 4000 m) which we approximate by a similar change Δh to h, smoothed out to a nice pulse-like wave. The linearized model will describe what happens until, when the wave comes close to the beach, the wave height becomes a sizeable fraction of the depth. After that, the wave breaks and the full equations are necessary.

1. Small amplitude waves (linearized model)

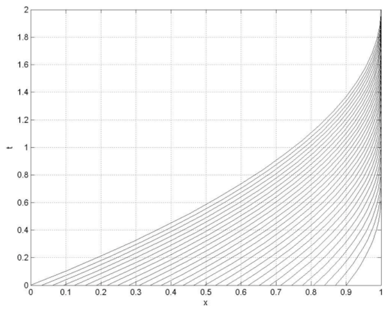
a) Supposing H constant and $h = h_0 + dh(x,t)$, $u = 0 + du(x,t)$, $du(x,0) = 0$ describe without detailed formulas what the wave pattern $dh(x,t)$



looks like after a while; what are the wave speeds? amplitudes? pulse lengths?

b) Choose initial conditions $dh(x,t = 0)$ and $du(x,t = 0)$ which will generate a *right*-running pulse only. *Hint*: Eigenvector of flux Jacobian.

Now let the sea floor be at $H(x) = (x/L-1)h_0$, $L = 100$ km, so the wave speeds change with x.
c) Where is the beach, $x = x_B$?
d) Consider and plot the *right*-running family of characteristics. Hint: They are NOT straight lines! (see right). Describe the wave shape with initial shape of 1b) at $x = 50$ km, 90km, 99 km. Hint: the fronts move along the characteristics, and thus the pulse length changes. The conservation law says



$$\frac{d}{dt}\int_{x=-\infty}^{x_B} h\,dx = 0$$ which determines the pulse height.

How far from the beach when the wave (in this approximation) is 2m high? 10m high?

2. The close-to-beach model
Use the full model from $x_L = x_B - d$ km (try $d = 10$, maybe as small as $d = 1$) to $x = x_B$ with a stepsize 10m. The initial data is the right running wave from 1d) at $x = x_B - d$. The boundary condition at x_L is $u = 0$. What is it at x_B? Try a reflecting wall there too. What really happens is that the wave runs past the steady shore-line, is dissipated by friction and turbulence and slowly trickles back into the sea.
2a) Run the Roe solver with $\Delta x = 20$m, 10m and 5m, Courant number as large as possible without instability, until the wave has been fully reflected at x_B. Plot the wave shapes when the front first hits x_B for the three stepsizes. Comments about order of accuracy, dissipation, dispersion?