Abstract framework

(i) Hilbert space V; with norm $\|\cdot\|_V$ and scalar product.

(ii) Bilinear form $a: V \times V \to \mathbb{R}$; determined by underlying differential equation.

(iii) Linear form $L: V \to \mathbb{R}$ determined by data.

We will formulate our differential equation using a bilinear and a linear form, and ~~using~~ search for a solution in the Hilbert space V.

A bilinear form $a(\cdot,\cdot)$ is a function taking values in $V \times V$ into $\mathbb{R}$. That is, $a(v,w) \in \mathbb{R}$ for all $v, w \in V$, such that $a(v,w)$ is linear in each argument:

$$a(\alpha_1 v_1 + \alpha_2 v_2, w_1) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1)$$

and $a(v_1, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_1, w_2)$

for all $\alpha_i \in \mathbb{R}$, $v_i, w_i \in V$.

A linear form $L(\cdot)$ is a function on V such that $L(v) \in \mathbb{R}$ for all $v \in V$, and linear in v: $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$