6

Ex: $\begin{cases} -\Delta u + u = f & \text{in } \Omega \subset \mathbb{R}^d \\ \partial_n u = 0 & \text{on } \Gamma \end{cases}$

Find variational formulation: mult. by v & integrate

$$\int_\Omega (-\Delta u + u) v \, dx = \int_\Omega (\nabla u \cdot \nabla v + uv) \, dx - \int_\Gamma \cancel{\partial_n u}^{\,0} v \, ds$$
$$= \int_\Omega f v \, dx$$

Find $u \in H^1(\Omega)$ s.t. $a(u,v) = L(v) \quad \forall v \in H^1(\Omega)$

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv) \, dx, \quad L(v) = \int_\Omega f v \, dx$$

Check if L-M applies: $V = H^1(\Omega)$.

(i) $a(\cdot,\cdot)$ V-elliptic?

$$a(v,v) = \int_\Omega (|\nabla v|^2 + v^2) \, dx = \|v\|^2_{H^1(\Omega)} = \|v\|^2_V$$

$\Rightarrow a(\cdot,\cdot)$ V-elliptic with $\alpha_1 = 1$

(ii) $a(\cdot,\cdot)$ continuous?

$$|a(v,w)| = \left|\int_\Omega (\nabla v \cdot \nabla w + vw) \, dx\right| \le \|\nabla v\| \, \|\nabla w\| + \|v\| \, \|w\|$$
$$= (\|\nabla v\| \;\; \|v\|) \cdot (\|\nabla w\| \;\; \|w\|)$$
$$\le (\|\nabla v\|^2 + \|v\|^2)^{1/2} (\|\nabla w\|^2 + \|w\|^2)^{1/2} = \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}$$
$$= \|v\|_V \|w\|_V \Rightarrow a(\cdot,\cdot) \text{ cont. with } \alpha_2 = 1$$

(iii) $L(\cdot)$ continuous?

$$|L(v)| = \left|\int_\Omega f v \, dx\right| \le \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)}$$
$$= \|f\|_{L_2(\Omega)} \|v\|_V \Rightarrow L(\cdot) \text{ cont. with } \alpha_3 = \|f\|_{L_2(\Omega)}$$

(i), (ii), (iii) $\Rightarrow$ L-M applies!