Ex: $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \Gamma \end{cases}$ (Poisson eqn. Dirichlet b. c.)

Find $u \in V$: $a(u,v) = L(v) \quad \forall v \in V$

$V = H^1_0(\Omega)$

$a(v,w) = \int_\Omega \nabla v \cdot \nabla w \, dx \qquad L(v) = \int_\Omega f v \, dx$

$$\|v\|^2_{H^1(\Omega)} = \|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)} \underset{\substack{\uparrow \\ \text{Poincaré-Friedrich ineq.}}}{\leq} (1+C)\|\nabla v\|^2_{L_2(\Omega)} = (1+C)a(v,v)$$

$\Rightarrow a(\cdot,\cdot)$ V-elliptic with $\kappa_1 = (1+C)^{-1} > 0$

$$a(v,w) = (\nabla v, \nabla w) \leq \|\nabla v\| \, \|\nabla w\| \leq \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} = \|w\|_V \|w\|_V$$

$\Rightarrow a(\cdot,\cdot)$ continuous with $\kappa_2 = 1$

$L(\cdot)$ continuous as before. $(\kappa_3 = \|f\|_{L_2(\Omega)})$

$\Rightarrow$ L-M applies $\Rightarrow$ existence & uniqueness!

Ex: $\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_n u = g & \text{on } \Gamma \end{cases}$

$V = H^1(\Omega)$, $a(u,v)$ as with $g=0$

$L(v) = \int_\Omega f v \, dx + \int_\Gamma g v \, ds$

$$|L(v)| \leq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)} \|v\|_{L_2(\Gamma)}$$

$$\leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)} + \|g\|_{L_2(\Gamma)} \|v\|_{H^1(\Omega)}$$

$\Rightarrow L(\cdot)$ cont. with $\kappa_3 = (\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)})$