Poincaré-Friedrichs ineq: There exist $c>0$ s.t. $\|\nabla v\|^2 \geq c\|v\|^2 \quad \forall v : v=0$ on Γ (and $v \in H^1(\Omega)$)

$f \neq 0$: mult by u & integrate:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = (f,u)$$

P.F. $\Rightarrow$ $\frac{1}{2}\frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq (f,u) \leq \|f\|\,\|u\| \leq \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|u\|^2$

$$\left(\begin{array}{l} ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2 \quad \forall a,b,\ \varepsilon>0 \\ \text{Proof: } 0 \leq (a-\varepsilon b)^2 = a^2 - 2\varepsilon ab + \varepsilon^2 b^2 \end{array} \right)$$

$$\Rightarrow \frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq \frac{1}{c}\|f\|^2$$

$$\Rightarrow \frac{d}{dt}\left(e^{ct}\|u\|^2\right) \leq \frac{e^{ct}}{c}\|f\|^2$$

$$\Rightarrow \|u(T)\|^2 \leq e^{-cT}\|u_0\|^2 + \frac{1}{c}\int_0^T e^{-c(T-t)}\|f\|^2 dt$$

With $f=0$: energy dissipates exponentially!