Wave equation

$$(1)\quad \begin{cases} \ddot{u} - \Delta u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \Gamma \times (0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dot{u}(x,0) = \dot{u}_0(x) & \text{in } \Omega \end{cases}$$

Write as first order system: $u_1 = \dot{u}$, $u_2 = u$

$$(2)\quad \begin{cases} \dot{u}_1 - \Delta u_2 = 0 & \text{in } \Omega \times (0,T] \\ -\Delta \dot{u}_2 + \Delta u_1 = 0 & \text{in } \Omega \times (0,T] \\ u_1(x,0) = \dot{u}_0(x) & \text{in } \Omega \\ u_2(x,0) = u_0(x) & \text{in } \Omega \\ u_2 = 0 & \text{on } \Gamma \times (0,T] \end{cases}$$

Mult. (1) by $\dot{u}$ & integrate over Ω: (set $f=0$)

$$(\ddot{u} - \Delta u, \dot{u}) = (\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) = (\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0 \Rightarrow \boxed{\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0}$$

Energy conservation!

Similar by mult. (2) by (u_1, u_2) & integrate $\Rightarrow$

$$(\dot{u}_1, u_1) - (\Delta u_2, u_1) + (-\Delta \dot{u}_2, u_2) + (\Delta u_1, u_2) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u_1\|^2 + \underbrace{(\nabla u_2, \nabla u_1)}_{=0} + (\nabla \dot{u}_2, \nabla u_2) - \underbrace{(\nabla u_1, \nabla u_2)}_{=0} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\|u_1\|^2 + \|\nabla u_2\|^2\right) = 0$$