Discretize Wave eqn. (2) by Crank-Nicolson $(\theta=\frac{1}{2})$

Find $(u_1^{m+1}, u_2^{m+1}) \in V_h \times V_h$ s.t.

$$\Rightarrow \left(\frac{u_1^{m+1}-u_1^{m}}{k}, v_1\right) + a\left(u_2^{m+\theta}, v_1\right)$$

$$+ \, a\left(\frac{u_2^{m+1}-u_2^{m}}{k}, v_2\right) - a\left(u_1^{m+\theta}, v_2\right) = 0 \quad \forall (v_1, v_2) \in V_h \times V_h$$

Investigate energy conservation for C-N:

Set $v_1 = u_1^{m+\theta}$, $v_2 = u_2^{m+\theta}$

$$\Rightarrow k\left(\theta-\frac{1}{2}\right)\left\|\frac{u_1^{m+1}-u_1^{m}}{k}\right\|^2 + \frac{\|u_1^{m+1}\|^2-\|u_1^{m}\|^2}{2k} + \cancelto{0}{a\left(u_2^{m+\theta}, u_1^{m+\theta}\right)}$$

$$+ \, k\left(\theta-\frac{1}{2}\right)\left\|\frac{\nabla(u_2^{m+1}-u_2^{m})}{k}\right\|^2 + \frac{\|\nabla u_2^{m+1}\|^2-\|\nabla u_2^{m}\|^2}{2k} - \cancelto{0}{a\left(u_1^{m+\theta}, u_2^{m+\theta}\right)} = 0$$

C-N: $\theta = \frac{1}{2} \Rightarrow$

$$\boxed{\|u_1^{m+1}\|^2 + \|\nabla u_2^{m+1}\|^2 = \|u_1^{m}\|^2 + \|\nabla u_2^{m}\|^2}$$

C-N energy conservative!