TMA371 Partial Differential Equations TM, 2000-04-25. Solutions

1. Consider the Dirichlet problem

$$-\nabla\cdot(a(x)\nabla u)=f(x),\quad x\in\Omega\subset\mathbb{R}^2,\qquad u=0,\ \text{for } x\in\partial\Omega.$$

Assume that c_0 and c_1 are constants such that $c_0\le a(x)\le c_1$, $\forall x\in\Omega$ and let $U=\sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate

$$||u-U||_{H_0^1(\Omega)}\le C\inf_{\chi\in M}||u-\chi||_{H_0^1(\Omega)}.$$

Solution: We formulate the continuous and approximate weak formulations as:

$$(1)\qquad (a\nabla u,\nabla v)=(f,v),\qquad \forall v\in H_0^1(\Omega),$$

and

$$(2)\qquad (a\nabla U,\nabla v)=(f,v),\qquad \forall v\in M,$$

respectively, so that

$$(3)\qquad (a\nabla(u-U),\nabla v)=0,\qquad \forall v\in M.$$

We may write

$$u-U=u-\chi+\chi-U,$$

where χ is an arbitrary element of M, it follows that

$$\begin{aligned}(a\nabla(u-U),\nabla(u-U)) &=(a\nabla(u-U),\nabla(u-\chi))\\ &\le ||a\nabla(u-U)||\cdot||u-\chi||_{H_0^1(\Omega)} \qquad (4)\\ &\le c_1||u-U||_{H_0^1(\Omega)}||u-\chi||_{H_0^1(\Omega)},\end{aligned}$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(5)\qquad (a\nabla(u-U),\nabla(u-U))\ge c_0||u-U||^2_{H_0^1(\Omega)}.$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)}\le\frac{c_1}{c_0}||u-\chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, and the proof is complete.

2. See the proof of Theorem 14.1 in the book.

3. Consider the boundary value problem

$$\begin{cases}-\Delta u+u=f, & x\in\Omega\subset\mathbb{R}^d,\\ n\cdot\nabla u=g, & \text{on}\quad \Gamma:=\partial\Omega,\end{cases}$$

where n is the outward unit normal to Γ.

(a) Show the following stability estimate: for some constant C,

$$||\nabla u||^2_{L_2(\Omega)}+||u||^2_{L_2(\Omega)}\le C(||f||^2_{L_2(\Omega)}+||g||^2_{L_2(\Gamma)}).$$

(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega=[0,1]$, $f(x)=1$, $g(0)=3$ and $g(1)=0$.

Solution: a) Multiplying the equation by u and performing partial integration we get

$$\int_\Omega \nabla u\cdot\nabla u + uu - \int_\Gamma n\cdot\nabla u u = \int_\Omega fu,$$

i.e.,

$$\|\nabla u\|^2+\|u\|^2=\int_\Omega fu+\int_\Gamma gu\le \|f\|\|u\|+\|g\|_\Gamma C_\Omega(\|\nabla u\|+\|u\|) \tag{6}$$

where $\|\cdot\|=\|\cdot\|_{L_2(\Omega)}$ and we have used the inequality $\|u\|\le C_\Omega(\|\nabla u\|+\|u\|)$. Further using the inequality $ab\le a^2+b^2/4$ we have

$$\|\nabla u\|^2+\|u\|^2\le\|f\|^2+\frac{1}{4}\|u\|^2+C\|g\|_\Gamma^2+\frac{1}{4}\|\nabla u\|^2+\frac{1}{4}\|u\|^2$$

which gives the desired inequality.
b) Consider the variational formulation

$$\int_\Omega \nabla u\cdot\nabla v+uv=\int_\Omega fv+\int_\Gamma gv, \tag{7}$$

set $U(x)=\sum U_j\psi_j(x)$ and $v=\psi_i$ in (7) to obtain

$$\sum_{j=1}^N U_j\int_\Omega \nabla\psi_j\cdot\nabla\psi_i+\psi_j\psi_i=\int_\Omega f\psi_i+\int_\Gamma g\psi_i,\quad i=1,\dots,N.$$

This gives $AU=b$ where $U=(U_1,\dots,U_N)^T,\ b=(b_i)$ with the elements

$$b_i=h,\ i=2,\dots,N-1,\quad b(N)=h/2,\quad b(1)=h/2+3,$$

and $A=(a_{ij})$ with the elements

$$a_{ij}=\begin{cases}-1/h+h/6, & \text{for } i=j+1 \text{ and } i=j-1\\ 2/h+2h/3, & \text{for } i=j \text{ and } i=2,\dots,N-1\\ 0, & \text{else.}\end{cases}$$

4. Consider the initial-boundary value problem

$$\begin{cases}\dot u-\Delta u=0, & x\in\Omega, \quad t>0,\\ u=0, & x\in\partial\Omega,\quad t>0,\\ u(x,0)=u_0(x), & x\in\Omega.\end{cases}$$

Show the stability estimates:

$$\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds\le\|u_0\|^2+C\int_0^t\|f(s)\|^2\,ds,$$

$$\|\nabla u(t)\|^2+\int_0^t\|\Delta u(s)\|^2\,ds\le\|\nabla u_0\|^2+C\int_0^t\|f(s)\|^2\,ds.$$

Solution: Multiplication by u gives

$$(\dot u,u)+\|\nabla u\|^2=(f,u)\le\|f\|\,\|u\|\le C\|f\|\,\|\nabla u\|\le\frac{1}{2}C\|f\|^2+\frac{1}{2}\|\nabla u\|^2.$$

Here $(\dot u,u)=\frac{1}{2}\frac{d}{dt}\|u\|^2$ and hence

$$\frac{d}{dt}\|u\|^2+\|\nabla u\|^2\le C\|f\|^2.$$

Integrating $\int_0^t \cdot ds$ gives the first inequality. To get the second one we multiply by $-\Delta u$:

$$(\dot{u}, -\Delta u) + ||\Delta u||^2 \le ||f||\,||\Delta u|| \le \frac{1}{2}||f||^2 + \frac{1}{2}||\Delta u||^2.$$

Here $(\dot{u}, -\Delta u) = \frac{1}{2}\frac{d}{dt}||\nabla u||^2$ and hence

$$\frac{d}{dt}||\nabla u||^2 + ||\Delta u||^2 \le ||f||^2.$$

5. Prove an a priori and an a posteriori error estimate for a finite element method for problem

$$-u'' + u' + u = f, \quad \text{in} \quad (0,1), \qquad u(0) = u(1) = 0.$$

Solution: Multiplication by $v \in H_0^1(I)$ and integration gives the variational formulation: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v + uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{8}$$

The corresponding finite element method $cG(1)$ is: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{9}$$

where

$$V_h^0 = \{v : v \text{ piecewise linear, continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Let now $e = u - U$, then (8)-(9) gives that

$$\int_I (e'v' + e'v + ev) = 0, \quad \forall v \in V_h^0. \tag{10}$$

We define the energy inner product and norm viz;

$$(v, w)_E = \int_I (v'w' + vw), \qquad ||v||_E^2 = (v.v)_E = \int_I [(v')^2 + v^2].$$

To derive **a posteriori** error estimates first we note that

$$\int_I e'e = \frac{1}{2}\int_0^1 \frac{d}{dx}(e^2) = 0, \qquad \text{since } e(0) = e(1) = 0. \tag{11}$$

Thus

$$\begin{aligned}
||e||_E^2 &= \int_I (e'e' + ee) = \int_I (e'e' + e'e + ee) \\
&= \int_I ((u-U)'e' + (u-U)'e + (u-U)e) = \{v = e \text{ in } (8)\} \\
&= \int_I fe - \int_I (U'e' + U'e + Ue) = \{v = \pi_h e \text{ in } (9)\} \\
&= \int_I f(e - \pi_h e) - \int_I (U'(e-\pi_h e)' + U'(e-\pi_h e) + U(e-\pi_h e)) \\
&= \{\text{partial integration on each subinterval}\} = \int_I R(U)(e - \pi_h e),
\end{aligned}$$

where $R(U) := f + U'' - U' - U = f - U' - U$, in each subinterval. Let now $||\cdot|| = ||\cdot||_{L_2(I)}$, then

$$||e||_E^2 \leq ||hR(U)||\,||h^{-1}(e - \pi_h e)|| \\ \leq C_i||hR(U)||\,||e'|| \leq C_i||hR(U)||\,||e||_E,$$

and hence

$$||e||_E \leq C_i||hR(U)||.$$

As for the **a priori** error estimate, a similar approach would result

$$\begin{aligned} ||e||_E^2 &= \int_I (e'e' + ee) = (11) = \int_I (e'e' + e'e + ee) \\ &= \int_I (e'(u-U)' + e'(u-U) + e(u-U)) = \{v = U - \pi_h u \text{ in } (10)\} \\ &= \int_I (e'(u-\pi_h u)' + e'(u-\pi_h u) + e(u-\pi_h u)) \\ &\leq ||e'||\,||(u-\pi_h u)'|| + ||e'||\,||u-\pi_h u|| + ||e||\,||u-\pi_h u|| \\ &\leq ||e||_E\{||(u-\pi_h u)'|| + 2||u-\pi_h u||\} \\ &\leq C_i||e||_E\{||hu''|| + ||h^2 u''||\}, \end{aligned}$$

consequently

$$||e||_E \leq C_i\{||hu''|| + ||h^2 u''||\}.$$

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