TMA372/MAN660 Partial Differential Equations TM, E3, GU
2001-12-18. Solutions

1. a) Derive the fundamental solution for the initial value problem

$$\dot{u}(t) + a(t)u(t) = f(t), \quad 0 < t \le T, \qquad u(0) = u_0.$$

b) Prove the stability estimates

$$\begin{array}{llll} i) & a(t) \ge \alpha > 0 & \Longrightarrow & |u(t)| \le e^{-\alpha t}|u_0| + \frac{1}{\alpha}\Big(1 - e^{-\alpha t}\Big) \max_{0 \le s \le t} |f(s)| \\ ii) & a(t) \ge 0 & \Longrightarrow & |u(t)| \le |u_0| + \int_0^t |f(s)|\, ds. \end{array}$$

solution. See lecture notes (chapter 9).

2. Show that for $a(t) > 0$, and for $N = 1, 2, \dots$, the piecewise linear approximate solution U for the problem 1 satisfies the a posteriori error estimate

$$|u(t_N) - U_N| \le \max_{[0,t_N]} |k(\dot{U} + aU - f)|, \quad k = k_n, \text{ for } t_{n-1} < t \le t_n.$$

solution. See the course book; chapter 9 (a simple version of theorem 9.2).

3. Prove *a priori* and *a posteriori* error estimates, in the energy norm $||v||_E^2 = ||v'||^2 + a||v||^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + u'(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad a \ge 0.$$

Solution:
(a) The Variational formulation:
(Multiply the equation by $v \in V$, integrate by parts over $(0, 1)$ and use the boundary conditions.)

$$(1) \qquad \text{Find } u \in V: \int_0^1 u'v'\, dx + \int_0^1 u'v\, dx + \int_0^1 auv\, dx = \int_0^1 fv\, dx, \quad \forall v \in V.$$

cG(1):
(2)

$$\text{Find } U \in V_h: \int_0^1 U'v'\, dx + \int_0^1 U'v\, dx + \int_0^1 aUv\, dx = \int_0^1 fv\, dx, \quad \forall v \in V_h,$$

where

$$V_h := \{v : v \text{ is continuous piecewise linear in } (0,1),\ v(0) = v(1) = 0\}.$$

From (1)-(2), we find
The Galerkin orthogonality:

$$(3) \qquad \int_0^1 \Big((u-U)'v' + (u-U)'v + a(u-U)v\Big)\, dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v, w)_E = \int_0^1 (v'w' + avw)\, dx, \qquad \forall v, w \in V.$$

A priori error estimate:

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 (e'e' + aee)\,dx = \Big\{ \int_0^1 e'e\,dx = \frac{1}{2}\int_0^1 \frac{d}{dx}\Big(e^2\Big)\,dx = \frac{1}{2}[e^2]_0^1 = 0 \Big\} \\
&= \int_0^1 (e'e' + e'e + aee)\,dx = \int_0^1 \Big(e'(u-U)' + e'(u-U) + ae(u-U)\Big)\,dx \\
&= \Big\{ v \in V_h \Big\} = \int_0^1 \Big(e'(u-v)' + e'(u-v) + ae(u-v)\Big)\,dx \\
&\quad + \int_0^1 \Big(e'(v-U)' + e'(v-U) + ae(v-U)\Big)\,dx = \Big\{(3)\Big\} \\
&= \int_0^1 \Big(e'(u-v)' + e'(u-v) + ae(u-v)\Big)\,dx \\
&= \int_0^1 \Big(e'(u-v)' + ae(u-v) + e'(u-v)\Big)\,dx \\
&\le ||e'||_{L_2} \cdot ||(u-v)'||_{L_2} + a||e||_{L_2}||u-v||_{L_2} + ||e'||_{L_2} \cdot ||u-v||_{L_2} \\
&\le ||e'||_{L_2} \cdot ||(u-v)'||_{L_2} + ||e||_E \cdot ||u-v||_{L_2} \\
&\le ||e||_E \cdot ||u-v||_E + ||e||_E \cdot ||u-v||_{L_2} \\
&\le ||e||_E \Big(||u-v||_E + ||u-v||_{L_2} \Big).
\end{aligned}
$$

This gives the a priori error estimate:

$$
||e||_E \le ||u-v||_E + ||u-v||_{L_2}, \quad \forall v \in V_h.
$$

A posteriori error estimate: We have that

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 (e'e' + aee)\,dx = \int_0^1 (e'e' + e'e + aee)\,dx \\
&= \int_0^1 (u'e' + u'e + aue)\,dx - \int_0^1 (U'e' + U'e + aUe)\,dx.
\end{aligned}
$$

Thus using (1) we get

$$
||e||_E^2 = \int_0^1 fe\,dx - \int_0^1 (U'e' + U'e + aUe)\,dx, \tag{4}
$$

which by (2) can be written as

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 fe\,dx - \int_0^1 (U'e' + U'e + aUe)\,dx \\
&\quad + \int_0^1 \Big(U'(\Pi_h e)' + U'\Pi_h e + aU\Pi_h e\Big)\,dx - \int_0^1 f\Pi_h e\,dx.
\end{aligned}
$$

Observe that the last line above is identically 0. Adding up we have

$$\begin{aligned}
||e||_E^2 =& \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + U'(e-\Pi_h e) + aU(e-\Pi_h e)\Big)\,dx \\
=& \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (U'+aU)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\
=& \{\text{partial integration}\} \\
=& \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (U'+aU)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\
=& \int_0^1 (f+U''-U'-aU)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\
=& \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \le ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\
\le& C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \le C_i||hR(U)||_{L_2}\cdot||e||_E.
\end{aligned}$$

This gives the <u>a posteriori error estimate:</u>

$$||e||_E \le C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - U' - aU = f - U' - aU$, on (x_{i-1}, x_i), $i = 1, \dots, M+1$.

4. a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases} u(x) + v''(x) = f(x), & v(0) = v(1) = 0, \quad 0 < x < 1, \\ u''(x) - v(x) = 0, & u(0) = u(1) = 0, \end{cases}$$

and show how the approximate solution (U, V) can be computed from the load vector F, using mass- and stiffness matrises.

b) Derive stability estimates for u and v, in terms of f, (e.g., through multiplying the first equation by v and the second by u).

Solution: a) Multiplying the equations in the system by the test functions φ and ψ, with $\varphi = \psi = 0$ for $x = 0$ and $x = 1$, and integrating by parts gives that

$$\begin{cases} \int_0^1 (\varphi u - \varphi' v') = \int_0^1 \varphi f, \\ \int_0^1 (-\psi' u' - \psi v) = 0. \end{cases} \tag{5}$$

Partitioning of $[0,1]$ into subintervals (elements) $I_j = [x_{j-1}, x_j]$, $x_j = j/(m+1)$, the linear approximations $U(x) = \sum_{j=1}^m U_j\varphi_j(x)$ and $V(x) = \sum_{j=1}^m V_j\varphi_j(x)$, with $\varphi_j(x)$ s being the usual piecewise linear basis functions, the $cG(1)$ approximation of the above system (4) can be formulated as: Find the nodal values U_j and V_j such that

$$\begin{cases} \int_0^1 (\varphi_i U - \varphi_i' V') = \int_0^1 \varphi f, \quad i = 1, \dots, m, \\ \int_0^1 (-\varphi_i' U' - \varphi_i V) = 0, \quad i = 1, \dots, m. \end{cases} \tag{6}$$

This gives $2m$ equations with the $2m$ unkown nodal values $U = [U_1, \dots, U_m]^T$ and $V = [V_1, \dots, V_m]^T$, which can be written in the matrix form as

$$\begin{cases} MU - SV = F, \\ -SU - MV = 0, \end{cases} \tag{7}$$

where M and S are the usual, 3-diagonal, mass- and stiffness matrises, respectively: M has $2h/3$ diagonal elements and $h/6$ super and subdiagonal elements. Corrsesponding elements for S are $2/h$ diagonal elements and $-1/h$ sub and superdiagonal elements. All other elemts are zeros. F is the load vector with elements $\int_0^1 \varphi_i f$. From the second equation above we get that $V = -M^{-1}SU$, which inserting in the first equation gives $U = (M + SM^{-1}S)^{-1}F$.

b) Multiply the first equation by u and the second equation by $-v$, add the two resulting equations and integrate over $[0,1]$. By partial integration we have then

$$\int_0^1 u^2 + v^2 = \int_0^1 uf,$$

Using Cauchy-Schwartz inequality we get

$$||u||^2 + ||v||^2 = \int_0^1 uf \le ||u||||f|| \le \frac{1}{2}||u||^2 + \frac{1}{2}||f||^2$$

This gives that $||u|| \le ||f||$, and consequently even $||v|| \le ||f||$.
We could alternatively multiply the first equation by $-v$ and the second by $-u$, add the two resulting equations and integrate over $[0,1]$. By partial integration we have this time

$$\int_0^1 (u')^2 + (v')^2 = \int_0^1 (-v)f,$$

Using, first Poincare', and then the Cauchy-Schwartz inequality we get

$$||u'||^2 + ||v'||^2 \le ||v||||f|| \le ||v'||||f|| \le \frac{1}{2}||v'||^2 + \frac{1}{2}||f||^2,$$

so that we have now $||v'|| \le ||f||$, and consequently even $||u'|| \le ||f||$.
We could obviously continue in this manner and get basic stability estimates for the, e.g., moment $v = u''$, through

$$||u''|| = ||v|| \le ||f||,$$

and for v'':

$$||v''|| = ||f - u|| \le ||f|| + ||u|| \le 2||f||.$$

5. Consider the following *Schrödinger* equation

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + iu_2$. a) Show that the *toatal probability* $\int_\Omega |u|^2$ is time independent.

Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over Ω and consider the real part.

b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$, and give the relation between $||u||$ and $||\nabla u||$ for the corresponding eigenfunction u.

c) What is the optimal constant C (expressed in terms of smallest eigenvalue λ_1), for which the inequality $||u|| \leq C||\nabla u||$ can fullfil for all functions u, such that $u = 0$ on $\partial\Omega$?

Solution: a) We multiply the shrödinger equation by $\bar{u}$ and integrate over Ω to obtain

$$\int_\Omega \bar{u}\dot{u} + i\int_\Omega \bar{u}\nabla u = \int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) + i\int_\Omega (u_1\dot{u}_2 - u_2\dot{u}_1 - \nabla\bar{u}\cdot\nabla u) = 0.$$

Now both real and imaginary part of the above expression is 0. Thus, considering the real part, we have

$$\int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (u_1^2 + u_2^2) = 0,$$

therefore $\int_\Omega |u|^2$ is independent of the time.

b) Multiplying the eigenvalue equation $-\Delta u = \lambda u$ by u, integrating over Ω, and using partial integration we get

$$\lambda\int_\Omega u^2 = \int_\Omega u(-\Delta u) = \int_\Omega |\nabla u|^2,$$

which gives $\lambda \geq 0$ (and also $\lambda > 0$, for $u \neq 0$). Further $||u|| = \frac{1}{\sqrt{\lambda}}||\nabla u||$. This indicates that the constant in the estimate $||u|| \leq C||\nabla u||$, satisfying for all functions u with $u = 0$ on $\Gamma := \partial\Omega$, can not be smaller than $\frac{1}{\sqrt{\lambda_1}}$, with $\lambda_1 > 0$ being the smallest eigenvalue. As a matter of fact we have the inequality $||u|| \leq \frac{1}{\sqrt{\lambda_1}}||\nabla u||$, for all u with $u = 0$ on Γ. This is due to the fact that we can represent u in terms of orthogonal eigenfunctions both "with and without gradient", i.e. $\int_\Omega u_i u_j = \int_\Omega \nabla u_i \cdot \nabla u_j = 0$, for $i \neq j$.

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