********* Exercise 1 **********

Example 1

$$-(k\,u')' \;=\; f$$
$$u(0) = 0 \quad u'(1) = \gamma$$
Assume $k = x$, $f = x^2$ and $\gamma = -1/3$.

a) Solve the problem with Galerkin's method using an ansatz of a quadratic polynomial.

b) Solve the problem with Galerkin's method using an ansatz of a cubic polynomial.

c) Calculate the analytical solution u.

d) Solve the problem with Galerkin's method using an ansatz of a quadratic polynomial and changing the boundary conditions to $u(0) = 2$ and $u'(1) = -1/3$.

******* *Answers* ********

a) Derive a weak formulation of the differential equation

$$-(k\,u')' \;=\; f \;\Rightarrow\; \int_0^1 -(k\,u')'\;v\;dx \;=\; \int_0^1 f\;v\;dx$$

Partial integration gives

$$\begin{aligned}\int_0^1 -(k\,u')'\;v\;dx \;&=\Big[-(k\,u')\;v\;\Big]_0^1 + \int_0^1 k\,u'\,v'\;dx = \\ &= -k(1)\,u'(1)\;v(1)\;+k(0)\,u'(0)\;v(0)\;+\int_0^1 k\,u'\,v'\;dx = \\ &= -k(1)\,\gamma\;v(1)\;+0+\int_0^1 k\,u'\,v'\;dx\end{aligned}$$

Thus

$$\int_0^1 k\,u'\,v'\;dx = \int_0^1 f\;v\;dx + k(1)\,\gamma\;v(1)$$

The variational equation is thus

$$a(u, v) = L(v)$$

with

$$a(u, v) = \int_0^1 k\,u'\,v'\;dx \qquad \text{and} \qquad L(v) = \int_0^1 f\;v\;dx + k(1)\,\gamma\;v(1)$$

The ansatz should be a second degree polynomial ($\rightarrow\; U = c_0 + c_1x + c_2x^2$) which fulfills essential boundary conditions ($u(0) = 0 \;\rightarrow\; U(0) = 0$). Thus, $U = c_1\,x + c_2\,x^2$. From this U we have the basis functions $\phi_1 = x$ and $\phi_2 = x^2$ giving $U = c_1\phi_1(x) + c_2\phi_2(x)$.

The Galerkin method is: Given the ansatz $U(x) = \sum_{i=1}^{n} c_i \phi_i(x)$, solve the system of equations $a(U, \phi_i) = L(\phi_i)$ for all , $i = 1 \dots n$.

Here, the equations thus are $a(U, \phi_2) = L(\phi_2)$ and $a(U, \phi_2) = L(\phi_2)$. Since $a(u, v)$ is bilinear and symmetric we have

$$a(U, \phi_1) = a(c_1\phi_1 + c_2\phi_2, \phi_1) = c_1\, a(\phi_1, \phi_1) + c_2\, a(\phi_2, \phi_1) = c_1\, a(\phi_1, \phi_1) + c_2\, a(\phi_1, \phi_2) = L(\phi_1)$$

and

$$a(U, \phi_2) = a(c_1\phi_1 + c_2\phi_2, \phi_2) = c_1\, a(\phi_1, \phi_2) + c_2\, a(\phi_2, \phi_2) = c_1\, a(\phi_2, \phi_1) + c_2\, a(\phi_2, \phi_2) = L(\phi_2)$$

giving the linear system of equations

$$\begin{pmatrix} a(\phi_1, \phi_1) & a(\phi_1, \phi_2) \\ a(\phi_1, \phi_2) & a(\phi_2, \phi_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} L(\phi_1) \\ L(\phi_2) \end{pmatrix}$$

Calculations now give

$$\begin{aligned} a(\phi_1, \phi_1) &= \int_0^1 k\ (\phi_1')^2\ dx = \int_0^1 x \cdot 1^2\ dx = 1/2 \\ a(\phi_1, \phi_2) &= \int_0^1 k\, \phi_1'\, \phi_2'\ dx = \int_0^1 x \cdot 1 \cdot 2x\ dx = 2/3 \\ a(\phi_2, \phi_2) &= \int_0^1 k\ (\phi_2')^2\ dx = \int_0^1 x\, (2x)^2\ dx = 1 \end{aligned}$$

and

$$\begin{aligned} L(\phi_1) &= \int_0^1 f\, \phi_1\ dx + k(1)\, \gamma\, \phi_1(1) = \int_0^1 x^2\, x\ dx + 1 \cdot \gamma \cdot 1 = 1/4 + \gamma = -1/12 \\ L(\phi_2) &= \int_0^1 f\, \phi_2\ dx + k(1)\, \gamma\, \phi_2(1) = \int_0^1 x^2\, x^2\ dx + 1 \cdot \gamma \cdot 1^2 = 1/5 + \gamma = -2/15 \end{aligned}$$

since $\gamma = -1/3$. The solution then is

$$\begin{pmatrix} 1/2 & 2/3 \\ 2/3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/12 \\ -2/15 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/10 \\ -1/5 \end{pmatrix}$$

Thus $U(x) = x/10 - x^2/5$.

Note that $U'(x) = 1/10 - 2x/5 \ \Rightarrow\ U'(1) = --3/10 \neq -1/3$, i.e.not exactly fulfilled.

b) Use ansatz $U = c_1\, x + c_2\, x^2 + c_3\, x^3$.

c) $u = -x^3/9$. Since this is a cubic polynomial, the subspace chosen i b contains the solution, hence we find the correct one.
