Useful Calculus rules and inequalities

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Partial Integration:

$$\int_a^b v'(x)w(x)\ dx = [v(x)w(x)]_a^b - \int_a^b v(x)w'(x)\ dx$$

Green's formula in 1d:

$$\int_a^b v'(x)w'(x)\ dx = [v(x)w'(x)]_a^b - \int_a^b v(x)w''(x)\ dx$$

Green's formula in higher dimensions:

$$\int_\Omega \nabla v \cdot \nabla w\ dx = \int_{\partial\Omega} v\partial_n w\ ds - \int_\Omega v\Delta w\ dx,$$

where $\partial_n v = \nabla v \cdot n$, and n is the outward normal of the domain Ω.

Absolute values and integrals:

$$|\int_\Omega f(x)\ dx| \leq \int_\Omega |f(x)|\ dx$$

Cauchy-Schwarz inequality: For $f, g \in L_2(\Omega)$:

$$|(f,g)| \leq \|f\|_{L_2(\Omega)}\|g\|_{L_2(\Omega)}$$

Energy norm; Schwarz inequality: For a symmetric bilinear form $a(\cdot,\cdot) : V \times V \rightarrow \mathbb{R}$, we define the energy norm as

$$\|v\|_a \equiv a(v,v)^{1/2}$$

For the enrgy norm we have the Schwarz inequality:

$$|a(v,w)| \leq \|v\|_a \|w\|_a$$

Variant of Cauchy's inequality: For all positive real numbers $a, b, \epsilon > 0$:

$$ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$$

Proof: $0 \leq (a - \epsilon b)^2 = a^2 - 2\epsilon ab + \epsilon^2 b^2$. The inequality follows by moving $-2\epsilon ab$ to the left hand side, and then divide by 2ϵ.