**Final Exam (Re-exam)**
**Mathematical Models, Analysis and Simulation. DN2266.**
**Tuesday, June 7, 2011. 8-13.**

Name: ______________________________

Pers. No: ______________________________

Remember to show your work, motivate your answers well, and to clearly state the final answer. Partial credit will be given to partial solutions.

No books, notes or calculators are allowed. Good luck!

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|---|---|---|
| Problem 1: | ________ | (5) |
| Problem 2: | ________ | (6) |
| Problem 3: | ________ | (6) |
| Problem 4: | ________ | (6) |
| Problem 5: | ________ | (7) |
| Problem 6: | ________ | (4) |
| Problem 7: | ________ | (6) |
| **Total:** | ________ | (40) |

**1.** For each statement below, mark if it is true or false. (No motivation required). For a statement to be true, it needs to be true for all cases. Read each statement carefully.

a) If $\mathbf{A}$ is an $n \times n$ matrix with real eigenvalues $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_n$, then $A$ is diagonalizable.

b) The determinant of an orthogonal matrix is either 1 or $-1$.

c) If $\mathbf{Ax} = \mathbf{b}_1$ has no solution, $\mathbf{Ax} = \mathbf{b}_2$ has many solutions, then $\mathbf{Ax} = \mathbf{b}_1 + \mathbf{b}_2$ has many solutions.

d) Let $\mathbf{A}$ be an $m \times n$ matrix. If $m < n$ then all singular values of $\mathbf{A}$ are strictly positive.

e) If $\lambda_A$ is an eigenvalue of the matrix $\mathbf{A}$ and $\lambda_B$ is an eigenvalue of the matrix $\mathbf{B}$, then $\lambda_A + \lambda_B$ is an eigenvalue of $\mathbf{A} + \mathbf{B}$.

**2.** Least squares solution.

a) We want to find the line $y = c + dt$ that passes through four given points $(t_1, y_1)$,$(t_2, y_2)$,$(t_3, y_3)$ and $(t_4, y_4)$. Write down the system of equations in matrix form. If all four points fall on a straight line, then this system has a solution, otherwise not. Write down the equations for the least squares solution. What is the quantity that is minimized by the least squares solution?

b) We want to find the point $\mathbf{x}_P = (x_P, y_P)$ on the line $\Gamma$ given by $y = mx + b$ that is closest to a given point $\mathbf{x}_Q = (x_Q, y_Q)$. Show that the point $\mathbf{x}_P$ that is closest to $\mathbf{x}_Q$ is the point for which $\mathbf{x}_P - \mathbf{x}_Q$ is orthogonal to $\Gamma$.
*Hint:* Let $\hat{\mathbf{t}}$ be the unit vector with the same slope as $\Gamma$. Let $\bar{\mathbf{x}}$ be the point on $\Gamma$ such that $(\bar{\mathbf{x}} - \mathbf{x}_Q) \cdot \hat{\mathbf{t}} = 0$, and let $\mathbf{x}_P = \bar{\mathbf{x}} + \alpha\hat{\mathbf{t}}$. Show that the distance is minimized when $\alpha = 0$.

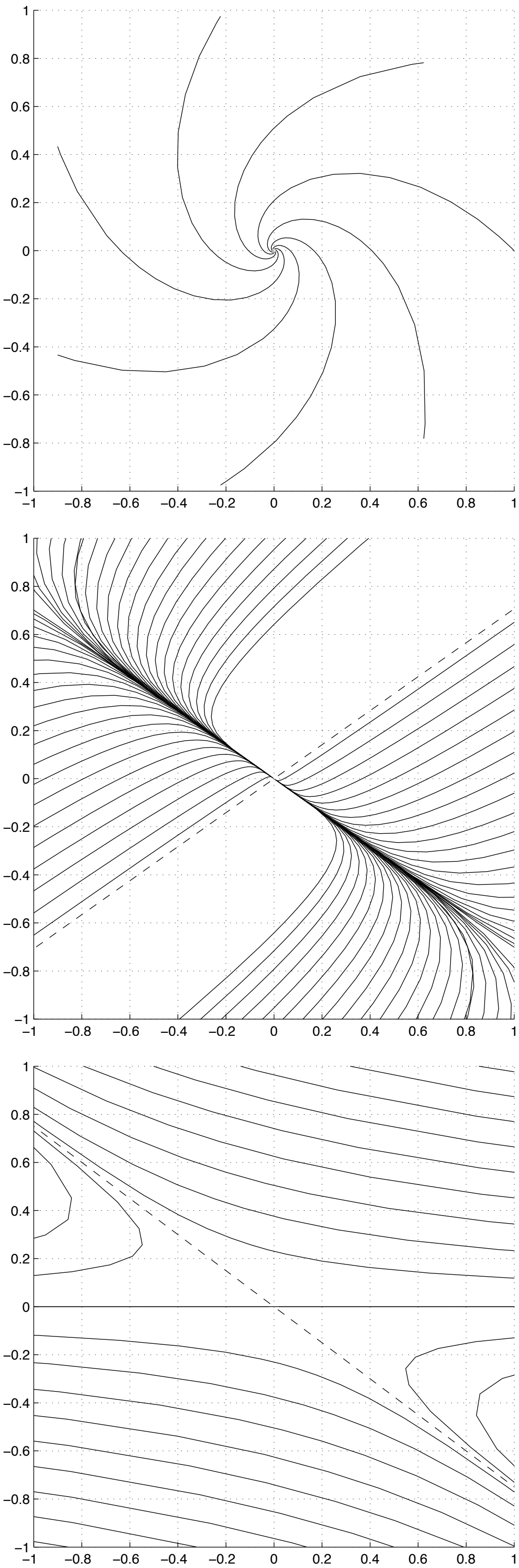
**3.** Consider $2 \times 2$ systems

$$\frac{d\mathbf{u}}{dt} = \mathbf{Au}. \quad (*)$$

where the general solution is given by

$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Below, we will consider phase portraits for such solutions. Note that there are no arrows included. For each of *a*)-*c*) below, make two sketches of the phase portrait, with different directions of the arrows. In each of the $3 \times 2 = 6$ cases, comment on what type of critical point the origin is, and whether it is stable or unstable. Say as much as you can about the eigenvalues and eigenvectors. If applicable, decide which is the fast and the slow manifold.




a)

b)

c)

**4.** Consider the predator-prey model

$$\begin{aligned}\frac{du}{dt} &= u(1-v-u)\\ \frac{dv}{dt} &= v(u-1/2-v).\end{aligned}$$

a) Which is the predator and which is the prey?

b) What are the critical points? Analyze the linearized equations around the strictly positive critical point. Using this result, what can you conclude for the stability of the critical point?

**5.** Consider

$$\begin{aligned} u_t &= cu_{xx} \qquad x \in (0,1), t > 0\\ u(x,0) &= g(x) \qquad\qquad\qquad\qquad (*)\\ u(0,t) &= u(1,t) = 0, \quad t > 0,\end{aligned}$$

where $c > 0$.

a) Show that

$$\int_0^1 (u(x,t))^2\,dx \le \int_0^1 (u(x,0))^2\,dx, \quad t > 0,$$

which is equivalent to

$$\|u(x,t)\|_{L^2} \le \|u(x,0)\|_{L^2}, \quad t > 0.$$

b) Use the result in $a$) to show that if $u(x,t)$ is a solution to (*), then it is unique.

c) Introduce $u_j^n = u(x_j, t_n)$, $x_j = j\Delta x$, $\Delta x = 1/N$, $j = 0, \ldots, N$ and $t_n = n\Delta t$, $n = 0, 1, \ldots$.
Consider the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right).$$

Which scheme is this? Using von Neumann analysis (assume periodic BC), show that this scheme is *unconditionally stable*. What does this mean?

d) State Lax equivalence theorem. Why is it useful?

**6.** The Chebyshev polynomials are defined on the interval $[-1, 1]$. Let $T_n(x)$, $n = 0, 1, \ldots$ denote the Chebyshev polynomial of degree $n$. The Chebyshev polynomials obey the following orthogonality relation:

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{1}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & m \neq n, \\ \pi & m = n = 0, \\ \frac{\pi}{2} & m = n \neq 0. \end{cases}$$

Give the expansion of a function $f(x)$ in the Chebyshev polynomials and give the formula for the coefficients in the expansion.

**7.** Consider the equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial^2 u}{\partial x^2} + f(x), \qquad && x \in [0, L], t > 0 \\ u(x, 0) &= g(x) && (*) \\ u(0, t) &= u(L, t), \quad t > 0, && \end{aligned}$$

where $c > 0$. The functions $f(x)$ and $g(x)$ are also $L$-periodic functions.

Let $u^N(x, t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{\frac{2\pi i k x}{L}}$ be a spectral expansion of $u$.

Derive the Galerkin equations for the expansion coefficients $\hat{u}_k(t)$, and give the initial conditions for $\hat{u}_k(0)$, $k = -N/2, \ldots, N/2 - 1$.