Final Exam
Mathematical Models, Analysis and Simulation. DN2266. Fall 2011.
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Tuesday, Dec 20, 2011. 8-13.

Remember to show your work, motivate your answers well, and to clearly state the final answer. Partial credit will be given to partial solutions.

1. (4p). For each statement below, mark if it is true or false. (No motivation required). For a statement to be true, it needs to be true for all cases. Read each statement carefully.

a) If the $n \times n$ matrix $\mathbf{A}$ is diagonalizable, then $\mathbf{A}$ is invertible.

b) If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal matrices, then $\mathbf{Q}_3 = \mathbf{Q}_1\mathbf{Q}_2$ is an orthogonal matrix.

c) Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of a square matrix $\mathbf{A}$.
If $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n > 0$ then $\mathbf{Ax} = \mathbf{b}$ has a unique solution, for any $n \times 1$ vector $\mathbf{b}$.

d) If the square matrix $\mathbf{A}$ is positive definite, then $\mathbf{A}^{-1}$ exists and is positive definite.

2. (8p). Here, we will consider two minimization problems.

a) Minimize $Q = \frac{1}{2}y_1^2 + \frac{1}{6}y_2^2$ subject to $y_1 + y_2 = 3$.

i) Write down the Lagrangian function including the Lagrange multiplier λ for this minimization problem.

ii) Derive the resulting linear system for (x, y, λ).

b) Now consider a minimization problem given by

$$Q = \mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y}, \tag{1}$$

with $\mathbf{K} \in \mathbb{R}^{2\times 2}$ a symmetric matrix, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and the (nonlinear) constraint

$$y_1^2 + y_2^2 = 1. \tag{2}$$

i) Formulate the problem given above in matrix-vector notation using a Lagrange multiplier, i.e. write down the Lagrangian in matrix-vector notation.

ii) Calculate the derivatives $\frac{\partial L}{\partial y_1}$, $\frac{\partial L}{\partial y_2}$, and $\frac{\partial L}{\partial \lambda}$ and derive the resulting (nonlinear) system.

iii) Assume now that the constraint (2) is satisfied. Write down the system of equations for $\frac{\partial L}{\partial y_1}$ and $\frac{\partial L}{\partial y_2}$ in matrix-vector notation. This formulation should look familiar to you. What kind of problem is it? What meaning does the constraint (2) have in this new problem? How does the minimum value of Q relate to the value of the Lagrange multiplier λ?

3. (8p). The equation of motion for the damped oscillations of a pendulum of mass m is given by

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0, \tag{3}$$

where g is the gravitational acceleration, a the length of the pendulum, and c is a positive constant. Here, x denotes the angle; at $x = 0$ the pendulum hangs straight down.

a) What are the critical points (if any) of (3)?

b) What are the conditions on the parameters c, m, q, a to get a stable *spiral*? What is the corresponding physical behaviour of the pendulum?

c) What are the conditions on the parameters c, m, g, a to get a stable *node*? What is the corresponding physical behaviour of the pendulum?

d) Let u_1, u_2 denote the first and second variable in the system formulation of (3). In the case when we have a stable node, what is the limit

$$\lim_{t\to\infty} \frac{u_2(t)}{u_1(t)}$$

(*Hint:* the limit will depend on the initial conditions.)

4. (8p). Consider

$$\begin{aligned} u_t + au_x &= -cu \qquad\qquad 0 \le x < \infty, t \ge 0, \\ u(x,0) &= g(x) \qquad\qquad\qquad\qquad\qquad\qquad (*) \\ u(0,t) &= f(t) \quad\; t > 0, \end{aligned}$$

where $a, c > 0$.

a) Write the characteristic equations ($\frac{dX}{dt} = \ldots, \frac{du(X(t),t)}{dt} = \ldots$). For which part of the positive quadrant $x > 0$, $t > 0$ is the solution completely determined by the *initial condition*? By the *boundary condition* at $x = 0$?

b) For this part, you can assume periodicity on $x \in [0, 1]$. Introduce $u_j^n = u(x_j, t_n)$, $x_j = j\Delta x$, $\Delta x = 1/N$, $j = 0, \ldots, N$ and $t_n = n\Delta t$, $n = 0, 1, \ldots$. Consider the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = -cu_i^n.$$

Apply von Neumann analysis and compute the growth/amplification factor.

i) Show that for $c = 0$, this scheme is always *unstable* for $\Delta t > 0$.

ii) For $c > 0$, derive the stability limit for Δt.

5. (8p). Korteweg and de Vries derived a model for waves in a long straight channel, such that only one space dimension remains. The equation for the elevation of the wave is

$$\frac{\partial \eta}{\partial t} = \sqrt{\frac{g}{d}} \frac{\partial}{\partial x} \left(d \cdot \eta + \frac{3}{4}\eta^2 + \frac{1}{2}\sigma \frac{\partial^2 \eta}{\partial x^2} \right), \quad -\infty < x < \infty, \quad t \geq 0,$$

where g is the gravitational constant, d the water depth, and $\sigma = d^3/3 - \mu d/(\rho g)$, where μ is the surface tension and ρ the water density.

a) Show that by a proper choice of length and time scales, L and T, $\eta = Lu$, $x = Lx'$ and $t = Tt'$, the equation becomes

$$\frac{\partial u}{\partial t'} = \frac{\partial}{\partial x'} \left(u + \frac{3}{4}u^2 + \frac{1}{2}a \frac{\partial^2 u}{\partial x'^2} \right).$$

What are L, T and a?

b) Applying yet another transformation, another more common form of the KdV equation is

$$u_t - 6uu_x + u_{xxx} = 0. \tag{4}$$

Consider this equation with periodic boundary conditions on $x \in [0, 1]$, and periodic initial conditions.
Let $u^N(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{\frac{2\pi i k x}{L}}$ be a spectral expansion of u.

i) Derive the Galerkin equations for the expansion coefficients $\hat{u}_k(t)$.

ii) Introduce a simple time-stepping scheme and describe a pseudo-spectral algorithm (i.e. without removing aliasing errors).

6. (4p). The discrete Fourier transform (DFT) for a real or complex function f is given by

$$f_j = \sum_{k=0}^{N-1} c_k e^{i2\pi jk/N}, \quad j = 0, \ldots, N-1$$

where the discrete Fourier coefficients are given by

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-i2\pi jk/N}, \quad k = 0, \ldots, N-1.$$

Now, assume that f and g are real functions with DFT coefficients c_k and d_k $(k = 0, \ldots, N-1)$, respectively. Introduce the complex function $\varphi = f + ig$ and let b_k, $k = 0, \ldots, N-1$, be the DFT coefficients of φ.

Show that

$$c_k = \frac{1}{2}\left(b_k + \bar{b}_{N-k}\right), \quad d_k = \frac{i}{2}\left(\bar{b}_{N-k} - b_k\right).$$

where $\bar{b}$ denotes the complex conjugate.

Hint: Start by expressing b_k and $\bar{b}_{N-k}$ in terms of c_k and d_k.