Lecture 5

Deductive systems

Deductive systems

The previous lectures have mainly been about the use of observations in science. This lecture will be concerned with the deductive side of science. The lecture is in three parts:

- A general discussion about formal systems.
- Paradoxes and impossibility theorems.
- The mathematization of the world.

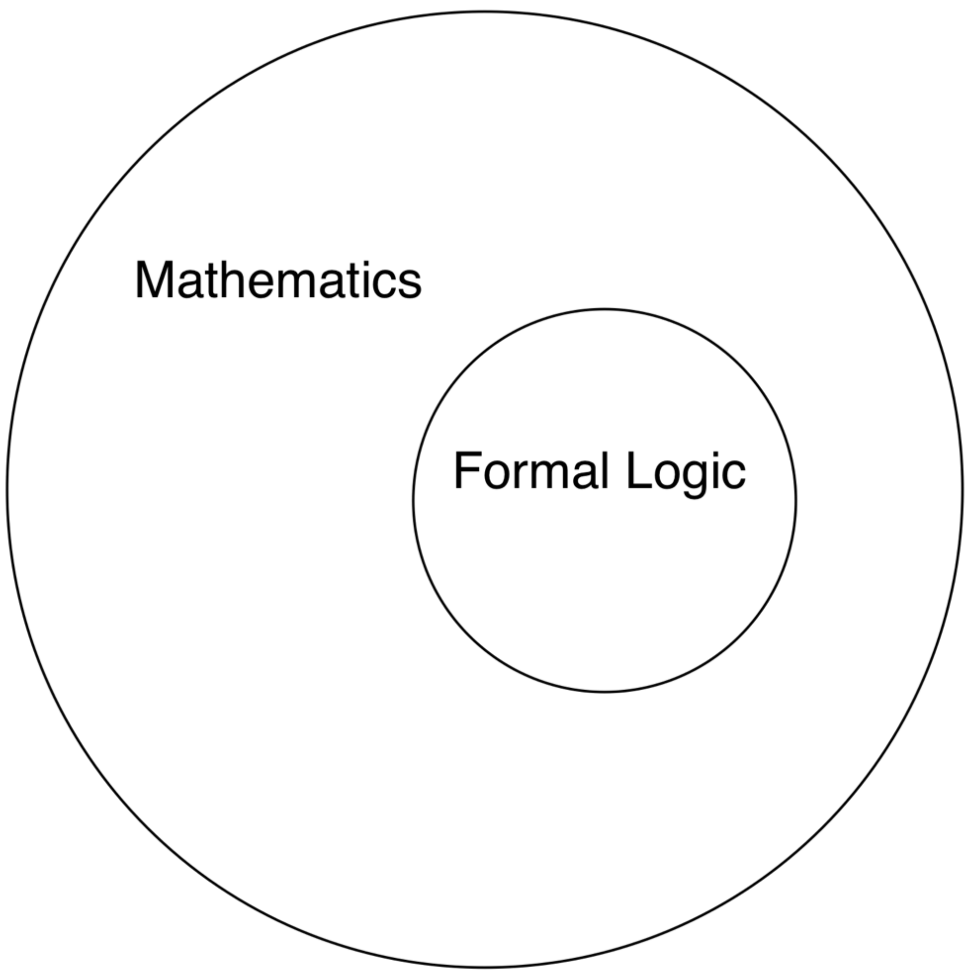
Users of formal systems

- Mathematicians - use them to prove mathematical theorems.
- Computer scientists - use them to design algorithms that solve problems.
- Philosophers - use them to define and analyze things.

Mathematics and Formal Logic

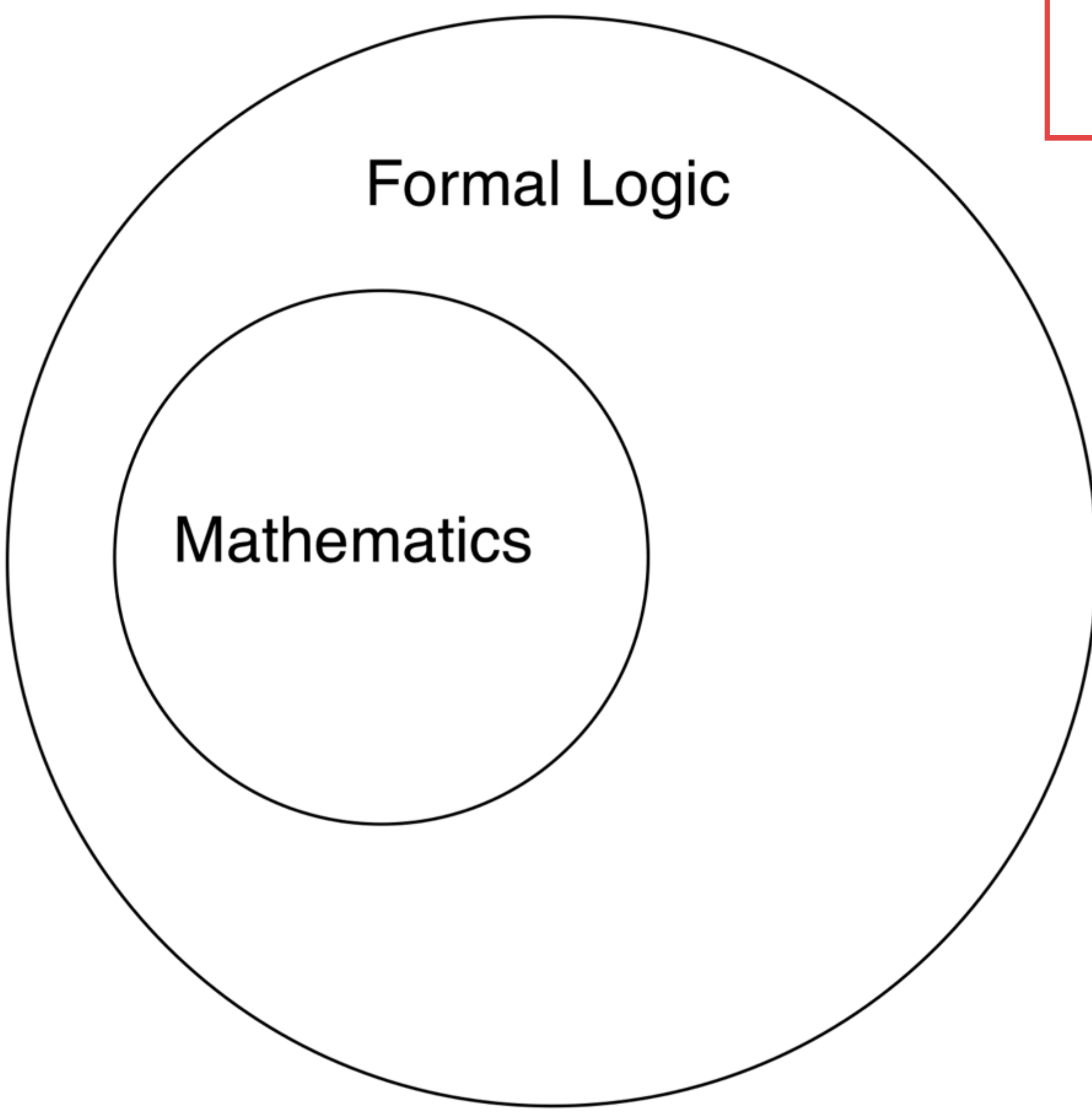
What is the connection between Mathematics and Formal Logic? Here are some suggestions:

Formal Logic is a part of Mathematics



This would probably be what mathematicians think

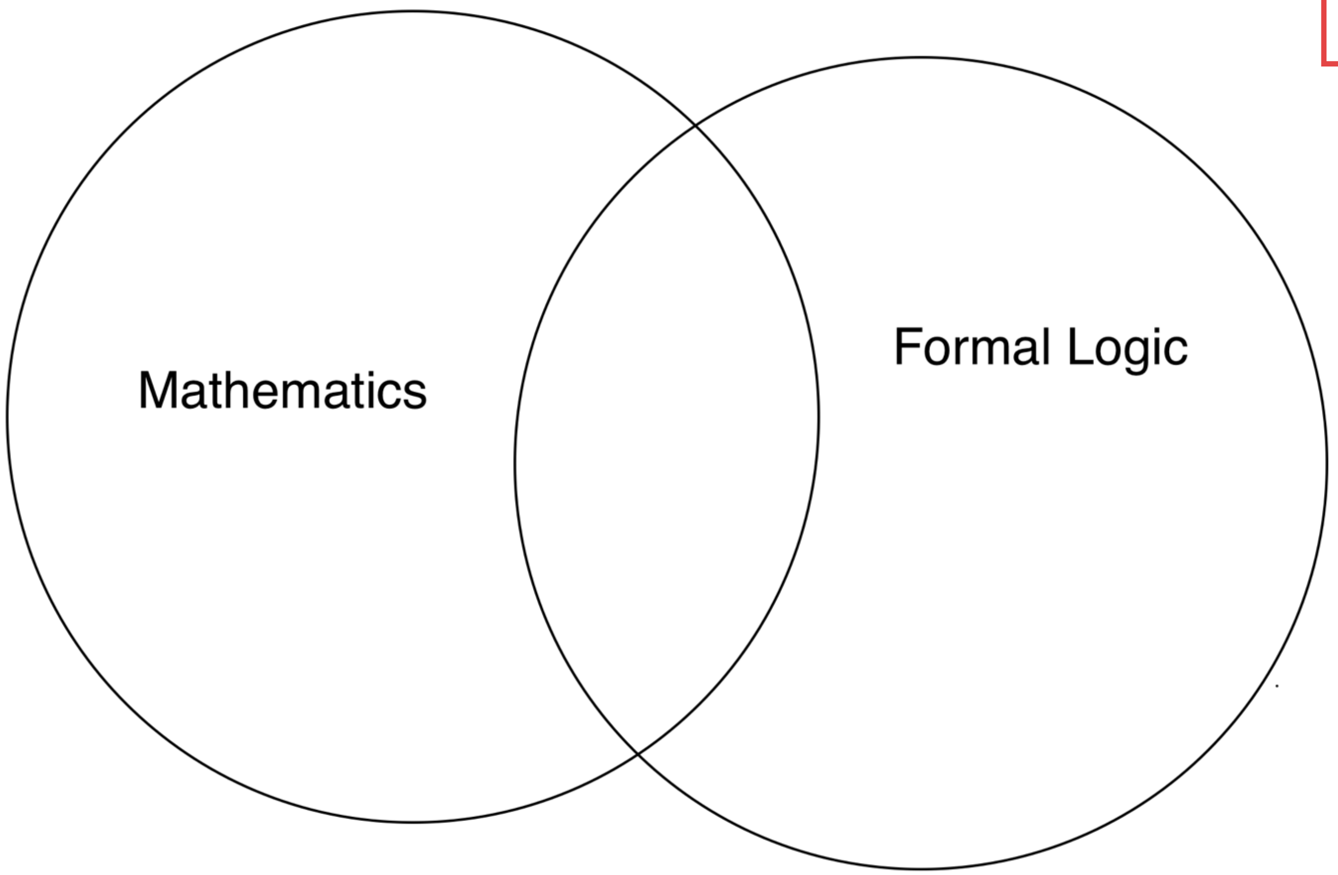
Mathematics is a part of Formal Logic



This is what the pioneers in Formal Logic thought

Neither is a part of the other

Nowadays, this seems to be natural view



Three components of a deductive system

- Vocabulary
- Deduction Rules
- Axioms

Vocabulary

We will look at some text from different disciplines all using formal syntax. It is normally rather easy to recognize the discipline.

Mathematics

Corollary 10.82 (Künneth Formula for Homology, II). *Let R be a right hereditary ring, let $(\mathbf{A}, \delta)$ be a complex of projective right R-modules, and let $\mathbf{C}$ be a complex of left R-modules.*

(i) *For all $n \geq 0$, there is an exact sequence*

$$\bigoplus_{p+q=n} H_p(\mathbf{A}) \otimes_R H_q(\mathbf{C}) \overset{\alpha}{\rightarrowtail} H_n(\mathbf{A} \otimes \mathbf{C}) \overset{\beta}{\twoheadrightarrow} \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(\mathbf{A}), H_q(\mathbf{C})),$$

where $\alpha_n \colon \sum_p \mathrm{cls}(b_p) \otimes c_{n-p} \mapsto \sum_p \mathrm{cls}(b_p \otimes c_{n-p})$, *and both λ_n and μ_n are natural.*

(ii) *For all $n \geq 0$, the exact sequence splits:*[8]

$$H_n(\mathbf{A} \otimes_R \mathbf{C}) \cong \Big[\bigoplus_{p+q=n} H_p(\mathbf{A}) \otimes_R H_q(\mathbf{C}) \Big] \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(\mathbf{A}), H_q(\mathbf{C})).$$

Theoretical physics

Schrödinger's Equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

i is the imaginary number, $\sqrt{-1}$.

$\hbar$ is Planck's constant divided by 2π: 1.05459×10^{-34} joule·second.

$\psi(\mathbf{r},t)$ is the wave function, defined over space and time.

m is the mass of the particle.

∇^2 is the Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Formal Logic

Ga	*1, 2, DM*
(~Fa → Ga)	*2-3, CD*
<u>~Ga</u>	***Hypothesis***
Fa	*1, 5, DM*
(~Ga → Fa)	*5-6, CD*
((~Fa → Ga) ^ (~Ga → Fa))	*4, 7, Conj*
((Fa v Ga) → ((~Fa → Ga) ^ (~Ga → Fa)))	*1-8, CD*
<u>((~Fa → Ga) ^ (~Ga → Fa))</u>	***Hypothesis***
<u>~(Fa v Ga)</u>	***Hypothesis***
~Fa	*11, DM*
(~Fa → Ga)	*10, Simp*
Ga	*12, 13, MP*
~Ga	*11, DM*

Computer Science

$$\frac{}{\rightsquigarrow a}\ \textsc{EtaR} \qquad \frac{}{x.(x \bullet k) \rightsquigarrow k}\ \textsc{EtaL}$$

$$\frac{}{\rightsquigarrow [k/\alpha]s}\ \textsc{BetaR} \qquad \frac{}{a \bullet x.(s) \rightsquigarrow [a/x]s}\ \textsc{BetaL}$$

$$\frac{}{\mathrm{not}[a] \rightsquigarrow a \bullet k}\ \textsc{BetaNeg} \qquad \frac{}{inl\ a \bullet [k, l] \rightsquigarrow a \bullet k}\ \textsc{BetaCoP}$$

$$\frac{}{, l] \rightsquigarrow a \bullet l}\ \textsc{BetaCoProd2} \qquad \frac{}{<a, b> \bullet \mathrm{fst}\ k \rightsquigarrow a \bullet k}\ \textsc{BetaPr}$$

$$\frac{}{\mathrm{snd}\ k \rightsquigarrow b \bullet k}\ \textsc{BetaProd2}$$

Chemistry

$$2\mathrm{NaX} + \mathrm{Ca}^{2+} \leftrightarrow \mathrm{CaX}_2 + 2\mathrm{Na}^+ \quad \log K_{\mathrm{NaCa}} = -0.20$$

$$K_{\mathrm{NaCa}} = \frac{\beta_{\mathrm{Ca}}[\mathrm{Na}^+]^2}{\beta_{\mathrm{Na}}^2 \gamma_0^2 [\mathrm{Ca}^+]} = 0.62$$

$$\mathrm{NaCl}_{(aq)} \leftrightarrow \mathrm{Na}^+ + \mathrm{Cl}^- \quad \log K_{\mathrm{NaCl}} = 0.78$$

$$K_{\mathrm{NaCl}} = \frac{\gamma_0^2 [\mathrm{Na}^+][\mathrm{Cl}^-]}{\left[\mathrm{NaCl}_{(aq)}\right]} = 6.0$$

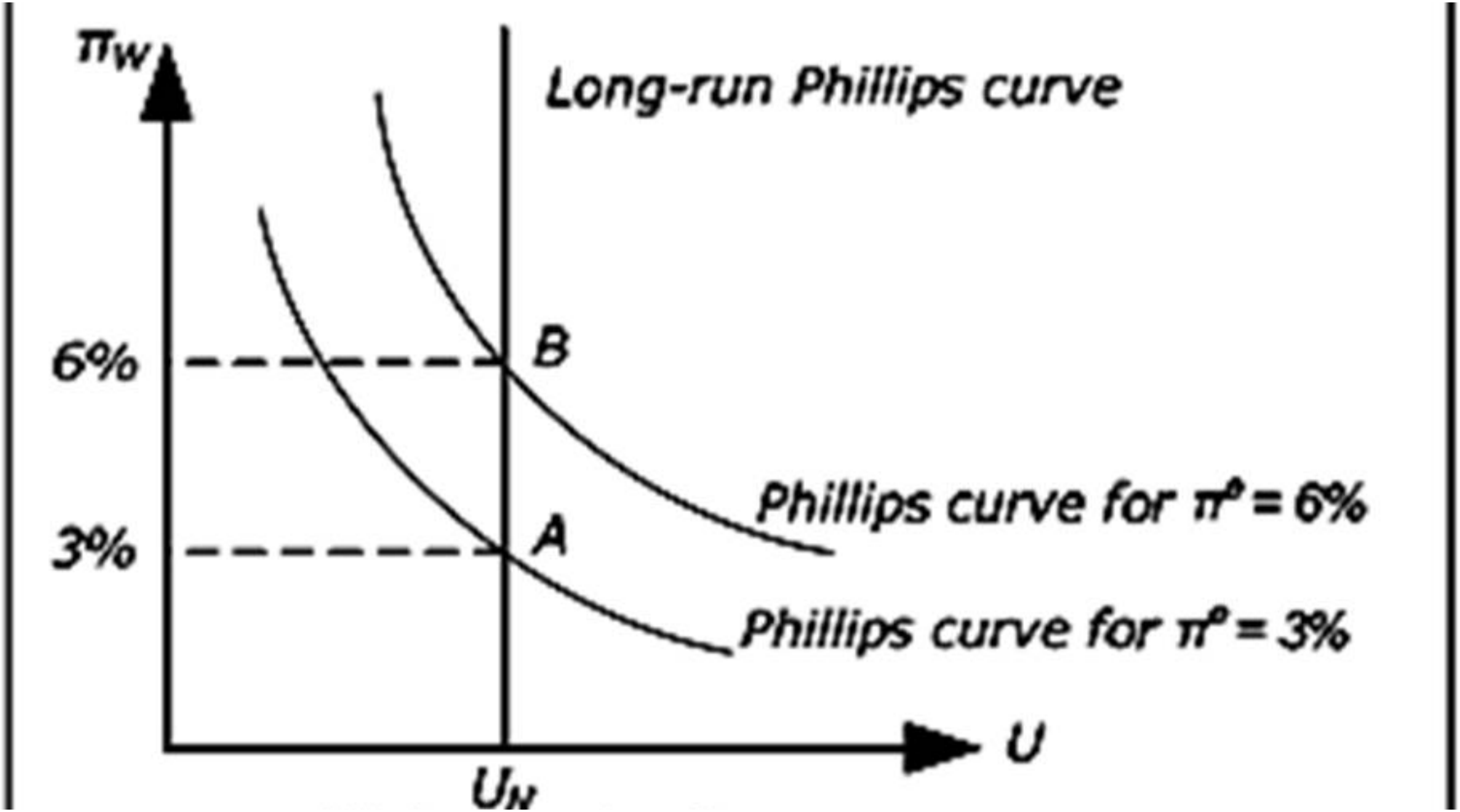
$$\mathrm{CaCl}^+ \leftrightarrow \mathrm{Ca}^{2+} + \mathrm{Cl}^- \quad \log K_{\mathrm{CaCl}} = 0.79$$

$$K_{\mathrm{CaCl}} = \frac{\gamma_0^4 [\mathrm{Ca}^{2+}][\mathrm{Cl}^-]}{[\mathrm{CaCl}^+]} = 5.0$$

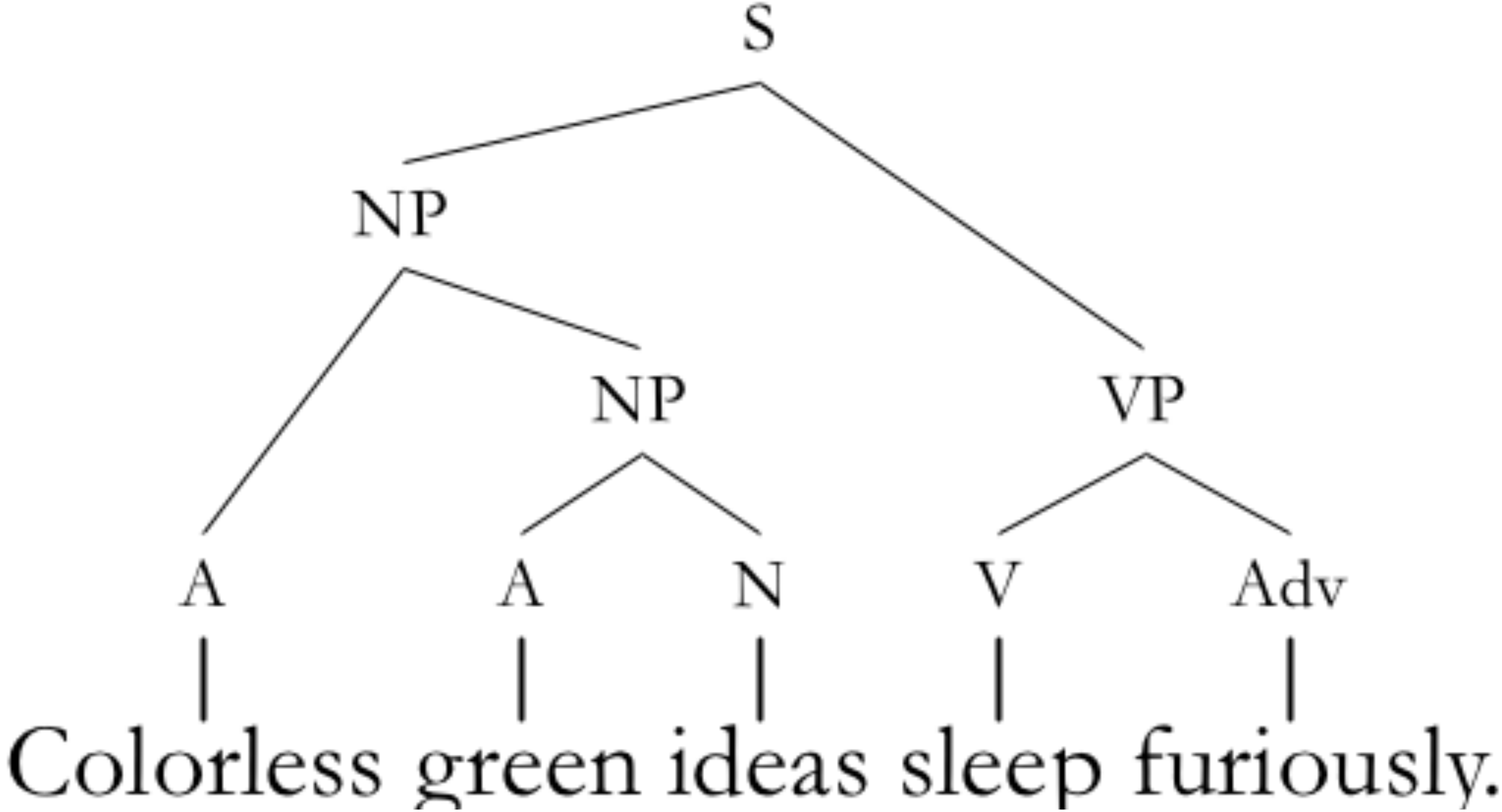
$$c_{\mathrm{Na}} = [\mathrm{Na}^+] + \left[\mathrm{NaCl}_{(aq)}\right] + c_{\mathrm{NaX}}$$

$$c_{\mathrm{Ca}} = [\mathrm{Ca}^+] + [\mathrm{CaCl}^+] + c_{\mathrm{CaX}_2}$$

Mathematical Economics



Linguistics



Vocabularies

- In a deductive system the vocabulary is roughly the syntax of the language we use in the system.
- Less formally, we can say that the vocabulary defines the type of expressions you can expect to find in the system.
- For instance, in text on evolutionary theory you would expect to find words like *natural selection* and so on.
- In formal logic the vocabulary is defined in a very precise way.

Deduction rules

- All deduction systems have some set of formal and informal rules which tells us what conclusions we can prove from other statements.
- In physics the rules are somewhat informal and established by praxis.
- In formal logic the deduction rules are where precisely defined.
- In mathematics it can happen that the deduction rules are implicitly understood. They can, however, be exactly stated (one would hope?)

Axioms

- The main idea is that the axioms are basic truths (intuitive truths maybe).
- Starting with axioms and using the deduction rules we create theorems.
- The axioms and theorems are the only truths in the system.
- In formal systems we divide the axioms into logical and non-logical axioms.
- In some systems with very strong deduction rules we have no logical axioms at all. Natural deduction is one example.

Do the axioms have to be true?

- The classic idea was that the axioms should be basic and fundamental truths.
- But later mathematicians realized that we could regard the axioms as assumptions and deduce consequences of these assumption.
- And important example of this is Non-Euclidian Geometries, developed in the 19th century.

Methodology?

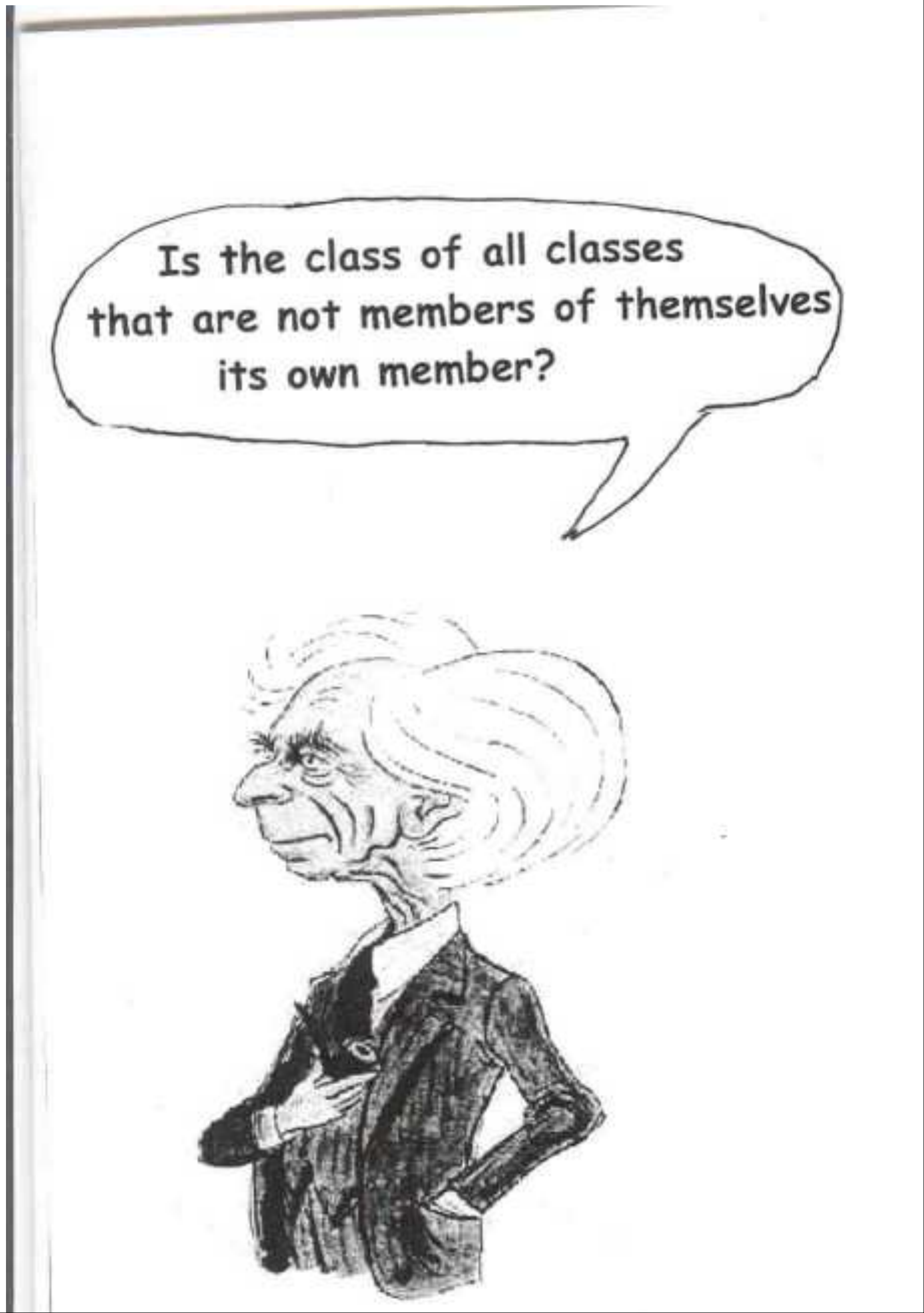
- It seems to be very hard to give prescriptions for how research with deductive methods should be done.
- Its not that hard to learn techniques for checking if proofs are correct. The difficult thing is to find good theorems and theories.
- This is essentially a *creative* activity. And there are no recipes for creativity.
- Or are there? The best way of learning how to find proofs is to imitate existing proofs.
- Some other tricks will be described in a later lecture.

Paradoxes and impossibility theorems

- We will give a brief discussion of some problems and paradoxes related to deductive systems and mathematics.
- We will describe two great crisis in the history of logic and mathematics

Russell's paradox

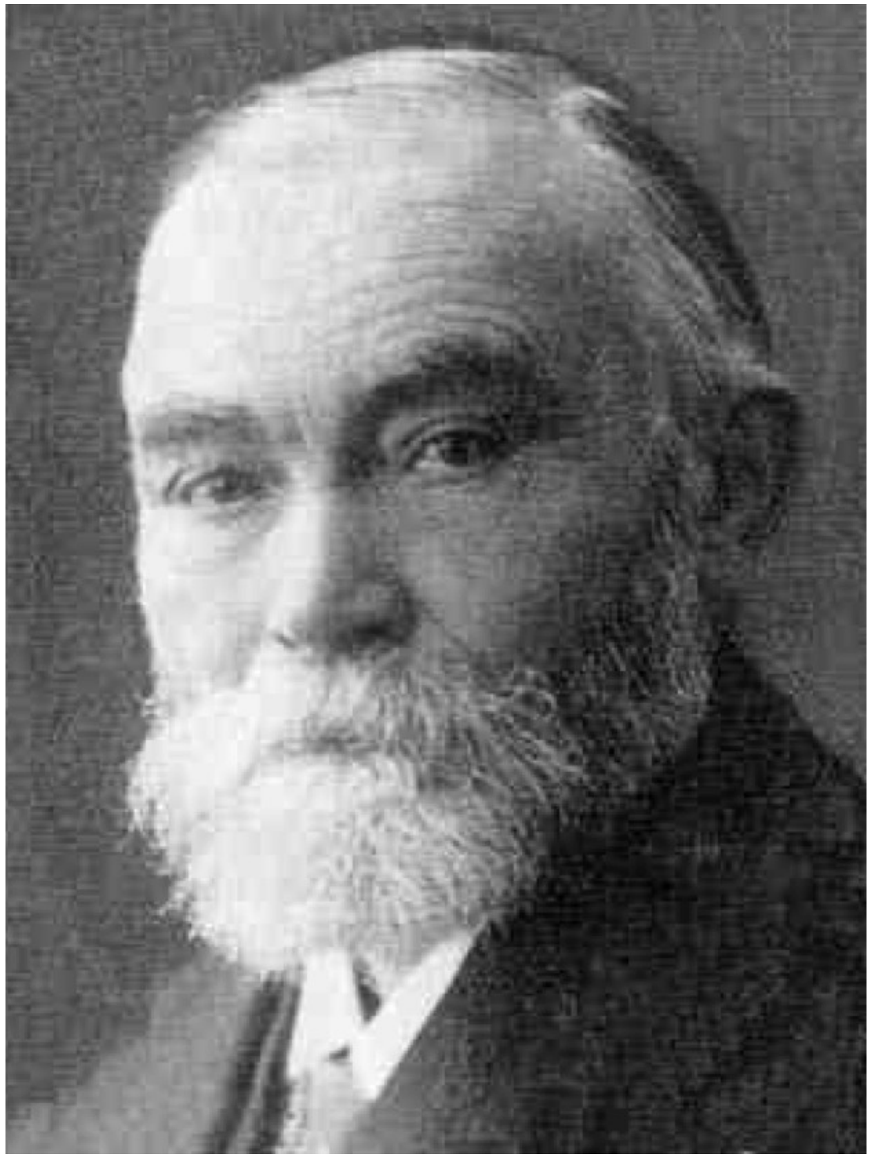
- The first crisis was in the early 20th century.
- We will start with some history.



Frege and mathematical logic

- Gottlob Frege created the modern mathematical logic at the end of the 19th century.
- He tried to construct all mathematics with logic.
- The starting point was a formalized version of set theory.
- Among other things Frege postulated that if P(x) is any predicate there always exists a set of all objects x such that P(x) is true:

$$\{x{:}P(x)\}$$



Bertrand Russell

- In the beginning of the 20th century Russell showed that Frege's axiom leads to contradictions.

- If we define

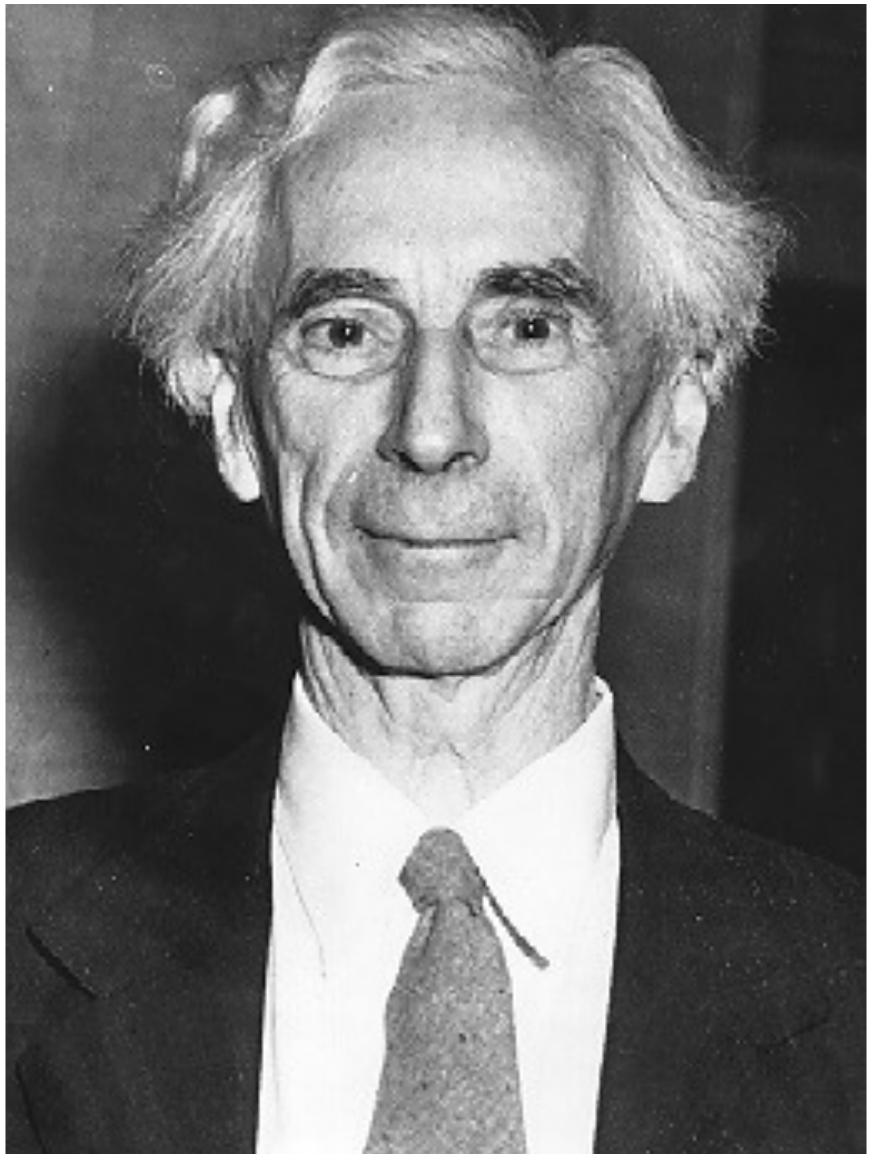
$$P(x)\colon x \notin x$$

And $$M = \{x \colon x \notin x\}$$

What happens then? Is $M \in M$

or $M \notin M$

true?



Some related paradoxes

- The liar paradox - 'I am lying'. True or false?
- Grelling paradox - Among English adjectives there are some, such as 'short', 'polysyllable', 'English', which apply to themselves. Let us call such adjectives *autological;* all others are *heterological.* Thus 'long', 'monosyllable', 'green' are heterological. But what about 'heterological'? Is it heterological or not?
- Berry paradox - Consider the phrase "The smallest positive integer not definable in under eleven words". There must be such an integer (why?). But this integer is definable in ten words!

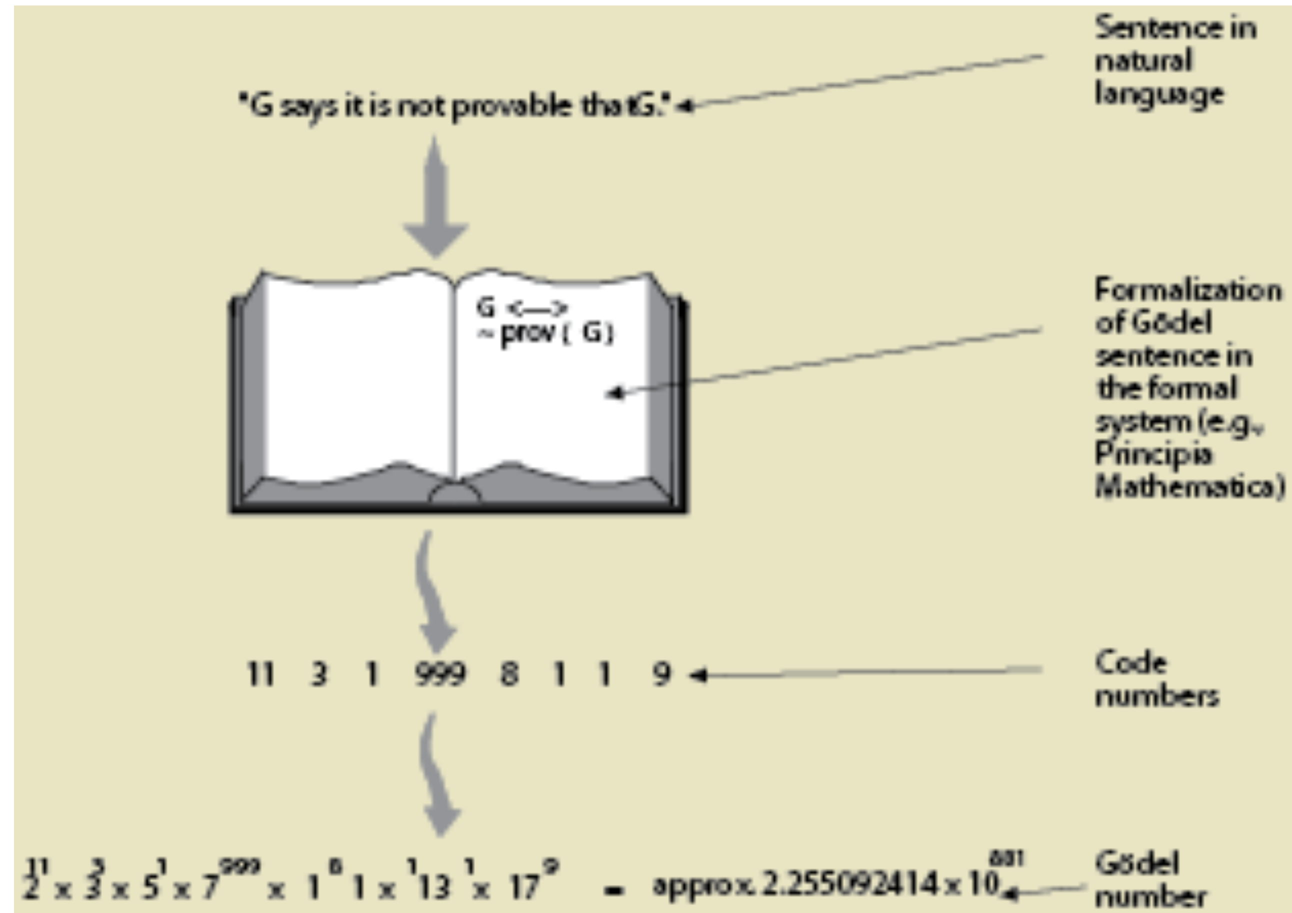
Russel's solution

- Russell found that Frege's axiom must be restricted in some way.
- His idea was to block the possibility that a set could be a member of itself.
- In order to do that he developed the so called *type theory* of sets.
- Other solutions came soon. The paradox is not considered a problem any more.
- But a disturbing fact is that Frege was one of the greatest logicians ever and he felt that his axiom was (intuitively) obvious. If he could make such a mistake, can we ever be certain that we don't make similar logical mistakes?

The ghost of self-reference

- Frege's problem was that an unexpected self-reference occurred.
- An analysis of the other paradoxes seem to show that the also are the victims of self-reference.
- Conjecture (the lecturer's): All 'paradoxes' are in some sense caused by self-reference.
- So if we just somehow can *block* all self-references there will be no paradoxes. Or ... ?

Gödel's Theorem



Gödel

- Kurt Gödel studied formal deductive systems of a special kind.

- He showed that all formulas in such a system can be given a so called *Gödel number.*

- He also showed that it is possible to construct a predicate that represents *provability.*

- Then he showed that there are sentences that cannot be proved in the system but still, in some more general sense, are true.

More details

- The Gödel Sentence: $G \leftrightarrow \neg Pr[S](\mathbf{G})$
- Gödel's theorem can be stated in at least two different forms.
- One form is that a sufficiently strong and (efficiently) decidable formal system must contain 'true' sentences which cannot be proved inside the system.
- Another form is that such a system must contain sentences which cannot be proved or disproved inside the system.
- To make things more complicated, there is a *Gödel's second incompleteness theorem* which says that the system cannot be proved to be consistent with methods inside the system.

Implications

- One thing Gödel's proof shows is that self-reference cannot actually be blocked. It is in a certain sense unavoidable.
- It also shows that the powers of formal systems are limited.
- We could of course accept these facts.
- Or we could just give up the idea of using formal systems.
- There are however some related theorems which are even more disturbing.

Tarski

- Alfred Tarski showed that the definition of truth is much more complicated than expected.
- The Tarski type of truth definition is like this: 'Snow is white' if and only if snow is white.
- This type of definition requires a *meta-level.* Truth comes in *layers,* so to say.
- And there is no way to define truth in any effectively decidable way.

Turing

- As we all know, Alan Turing defined the Turing Machine.
- He proved that there are natural problems which cannot be solved in an 'mechanical' way.
- An example is the halting problem.
- Another is the problem of finding proofs in first order logic.

So what are the conclusions of all this?

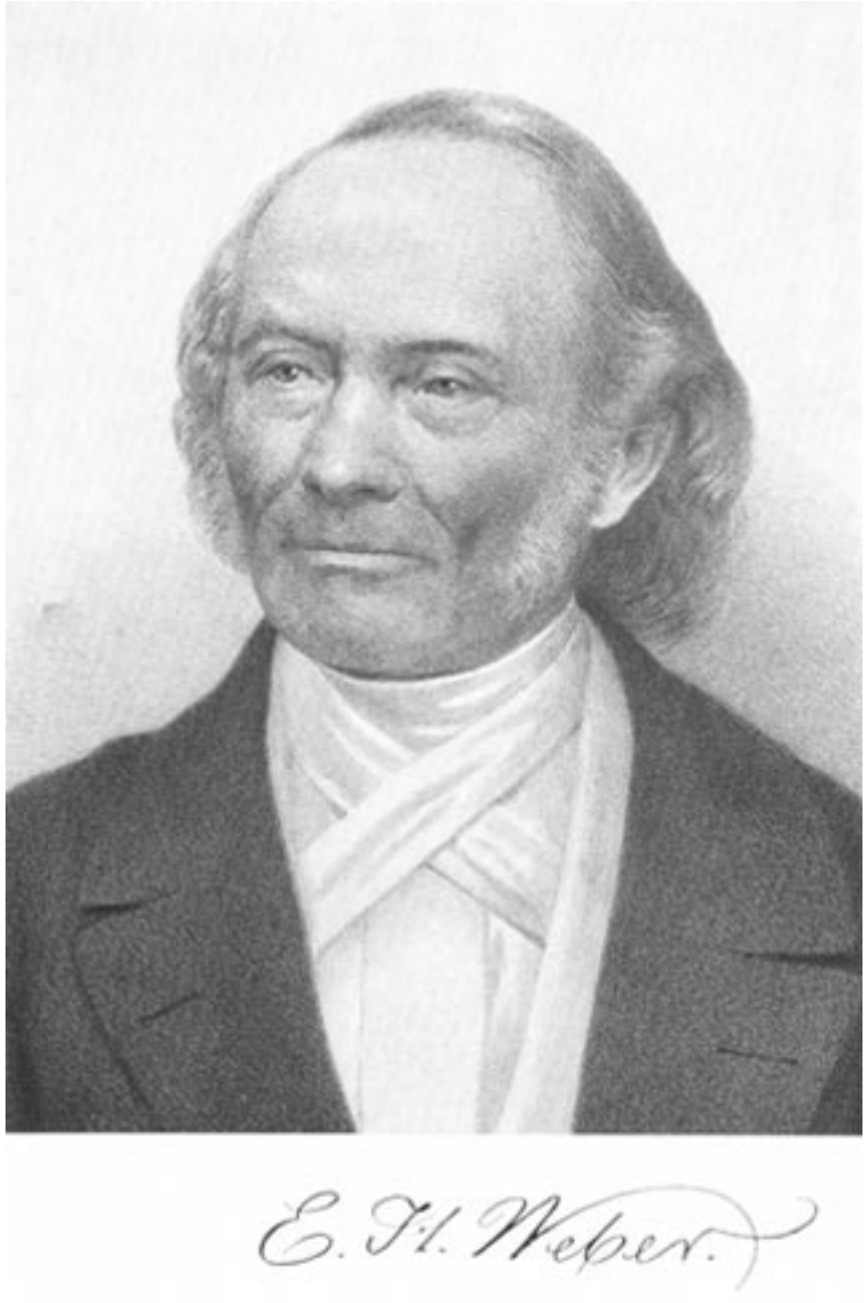
- Obviously, deductive systems cannot generate truths all in a mechanical way.
- We must sometime rely on other methods for finding truths. (But maybe still work inside a formal system?)
- Truth cannot be defined in a mechanical way.
- There are problems which cannot be solved in a mechanical way.
- But still, we will continue to use deductive systems.

The 'mathematization' of the world

- One of the reasons for the success of mathematics in science is the possibility of measuring things and then doing mathematical processing of the data.
- This fact can lead to the opinion that only measurable facts can be the subject of science.
- But in Social Sciences it is often claimed that *qualitative* data are as important as *quantitative.*
- We will illustrate how it is possible to use how it is possible to use mathematics to define measures on seemingly qualitative and subjective observations.

Weber's law

- Does 10 kg feel twice as 'heavy' as 5 kg?
- If we have a body with of weight m, can we find a function f(m) which measures the subjective 'heaviness' we experience?
- Yes, it is possible.



Weber's law II

- Let dm be the smallest change in mass that we (subjectively) can detect with our senses.
- In the beginning of the 19th century Weber showed that dm is linearly dependent on m, i.e. dm = cm for some constant m. (In an appropriate interval.)
- From this it is not hard to see that a natural definition of the subjective experience of 'heaviness' has the form k log m + w0 for some constants k, w0.

Utility theory

- Would you like to go to an interesting early morning lecture at KTH?
- Or would you rather sleep some hours more?
- Can you measure how much you want different things?
- John von Neumann suggested a way of measuring subjective preferences in an exact way.

Utility theory II

- In Game Theory and Mathematical Economics we make the assumption the we can personally order different things a, b, c ,... In preference order.
- We also make the assumption that we can measure how much we want different things by a utility function u.
- This means that we prefer e to q if and only if $u(e) > u(q)$.
- How is it possible to define such a function?

"What do you chose?
5000 $ or the secret box?"

- The idea von Neumann had was to imagine a virtual lottery. Let a be the thing we prefer most of everything and z the thing we prefer least of everything.
- Now, let L1 be a lottery with two possible outcomes: You get a or you get z. You get a with probability p and z with probability (1-p).
- And then, we take a thing k and imagine a trivial lottery L2 where you get k with certainty.
- Which one do you prefer? L1 or L2? It must depend on p.
- There should be a number p such that you are *indifferent* between L1 and L2. This p is the utility of k, i.e. $u(k) = p$.
- This means that $u(a) = 1$ and $u(z) = 0$.

Is it a realistic measure?

- To put it a little extreme, let us say that we want to compare *everything.*
- Let us then assume that a is "Happiness beyond all imagination" and z is "A horrible and painful death". Let k be "Attending a lecture in The Theory of Science".
- So L1 is the lottery [P(Happiness beyond all imagination) = p, P(A horrible and painful death) = (1-p)]
- And L2 is the lottery [Attending a lecture in The Theory of Science with certainty].
- For what p are you indifferent between L1 and L2?