Lecture 7

AntiRealism, Kuhn and some Mathematical Methodology

Strength and weakness of anti-realism

- Gives a certain intellectual sanitation.
- Is quite natural. The reality can never be exactly what we imagine it to be.
- At the same time, it seems that an anti-realist position can limit our ability to speak about things.

Realism vs. anti-realism

- A summary of the positions:
- Realists believe that science is an accurate description of reality, even those parts of it that cannot be observed directly.
- Anti-realists believe that science can only describe the observable parts of reality and that the theories often are only fictions or models about which we cannot say that they are true or false.
- What are the reasons for the different positions?

The "No miracles" – Argument

- This is an argument for realism.
- There are scientific theories that manages to describe the observable part of the reality very well.
- They do so by describing a model for a non-observable reality and explain how this is projecting on the observable reality.
- How do you explain the "miracle" that this description of the non-observable reality works so well?
- No miracle! It works because it is true!

Counter-arguments

- In the history of science, there are many examples of theories that explains observable data very well but still proved to be incorrect.
- One such example is The Phlogistone Theory. (It was observable data that ultimately led to the rejection.)
- A critical example is theories of light nature.

The argument from observability

- This is also an argument against anti-realism.
- Anti-realism is based on the supposed fact that we can divide the world into observable and non observable parts.
- But can we really do that in a consistent way?
- There are, for example. a gradual transition from observability with the eye to observability with electron microscopes. It is the first one a genuine observability but not the other one?

Counter-arguments

- That type of argument really just shows that observability is a vague concept. It does not necessarily mean that it is a meaningless concept.
- We can see that there are clear cases of what is observable and clear cases of things that are not. That's enough for anti-realism.

The argument from under-determination

- This is an argument for anti-realism.
- We imagine that we have a set of observed data. We want to find a theory that explains the data.
- It is possible to realize that there is always a variety of theories that may explain these data. The theories are being under-determined.
- If you are using a theory to explain the data, it is just an arbitrary tool for the explanation.
- That's exactly what anti-realists believe about theories.

Counter-arguments

- Although there are different theories that could explain the measured data, they are not all equivalent.
- It seems natural that there is some kind of selection criterion, for example, choosing the simplest theory.
- It also seems to be a lack of historically interesting examples of under-determination.

Laws

- What is a scientific law?
- It seems natural to interpret it as a regularity in nature.
- But there is a problem: The law of gravity specifies a rule for how bodies fall. It is not literarily true, however, due to air resistance. How can it then be a law?
- Laws should perhaps be interpreted as a tendency? They strike through, depending on strength.

The mystery of laws

- Why does nature follow laws?
- Does it do that?
- Newton's laws seems to be very successful.
- But is not the concept of force just *defined* in a way that makes it work?
- We may just see the laws that work?

Thomas Kuhn

Thomas Kuhn 1922-1996

- American. Doctor in physics at Harvard.
- Became more and more interested in the history and philosophy of science.
- In 1962 he published ”The Structure of Scientific Revolutions”. This is probably the most influential book on the philosophy of science ever published.
- The book introduced the phrase *paradigm shift.*

Kuhn's philosophy

- A paradigm consists of terms, methods, norms and ways of viewing thing. It defines our way of understanding the world (or at least a part of it).
- *Normal science* is science as it is done within the paradigm.
- In *revolutionary science* we reject the old paradigm and replaces it with a new one.

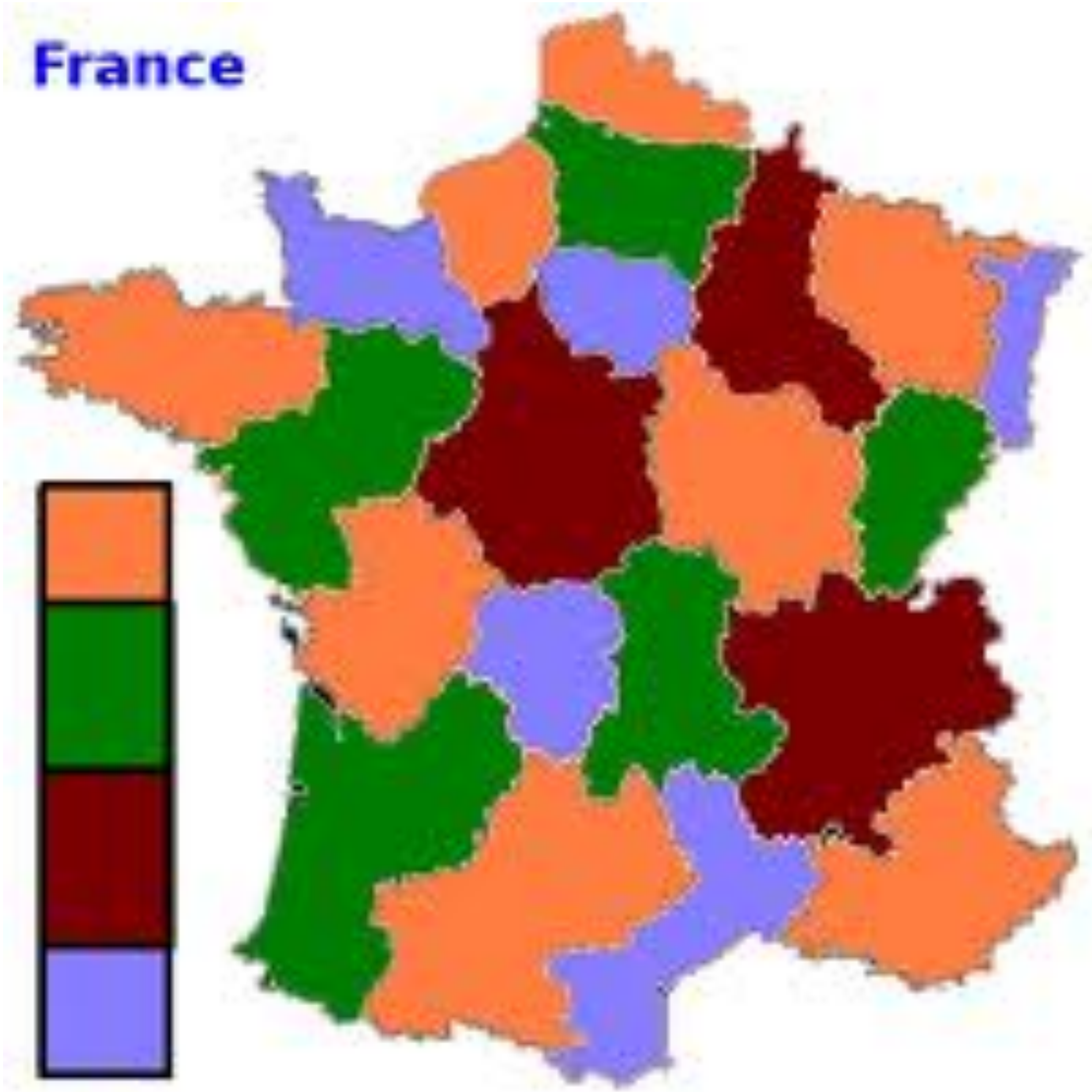
More details

- In normal science we don't put the paradigm on trial. All problems are handled within the paradigm.
- Within the paradigm we are doing ”puzzle-solving”. It is characteristic of real science that there is an established program for such problem solving.
- When a *crisis* occurs, it can lead to a paradigm shift.
- Such a shift is often done for *irrational* reasons.
- Two paradigms are *incommensurable* with each other.

Problems with Kuhn's philosophy

- Is it a recommendation for how science should be done?
- Yes, in a way. The philosophy focuses on the importance of stability in normal science.
- We would like to think that a paradigm shift always leads to a *better* paradigm. How can we tell if this is actually the case?
- Kuhn doesn't provide a clear answer to this question.

A case study: The Four-colour Theorem

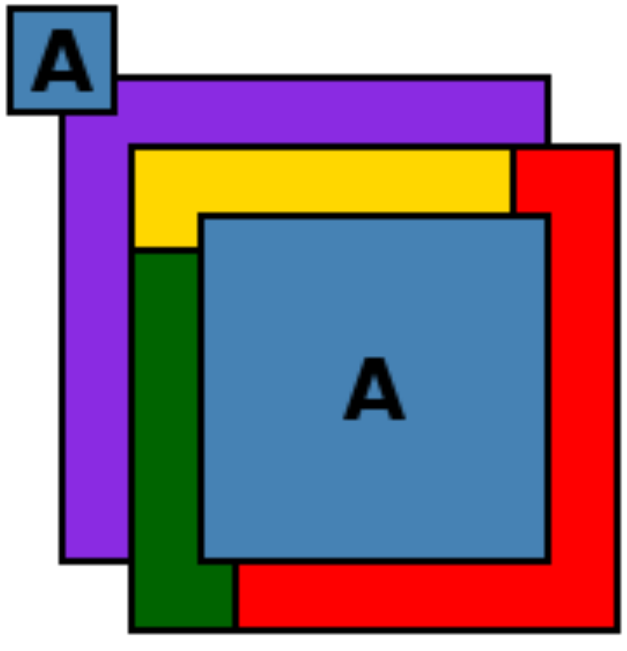


The theorem

- Every (planar) map can be coloured with four colours. A colouring is required to be such that no neigbouring countries have the same colour.
- This theorem was conjectured in 1852 and finally proved in 1976.

Is the theorem true?

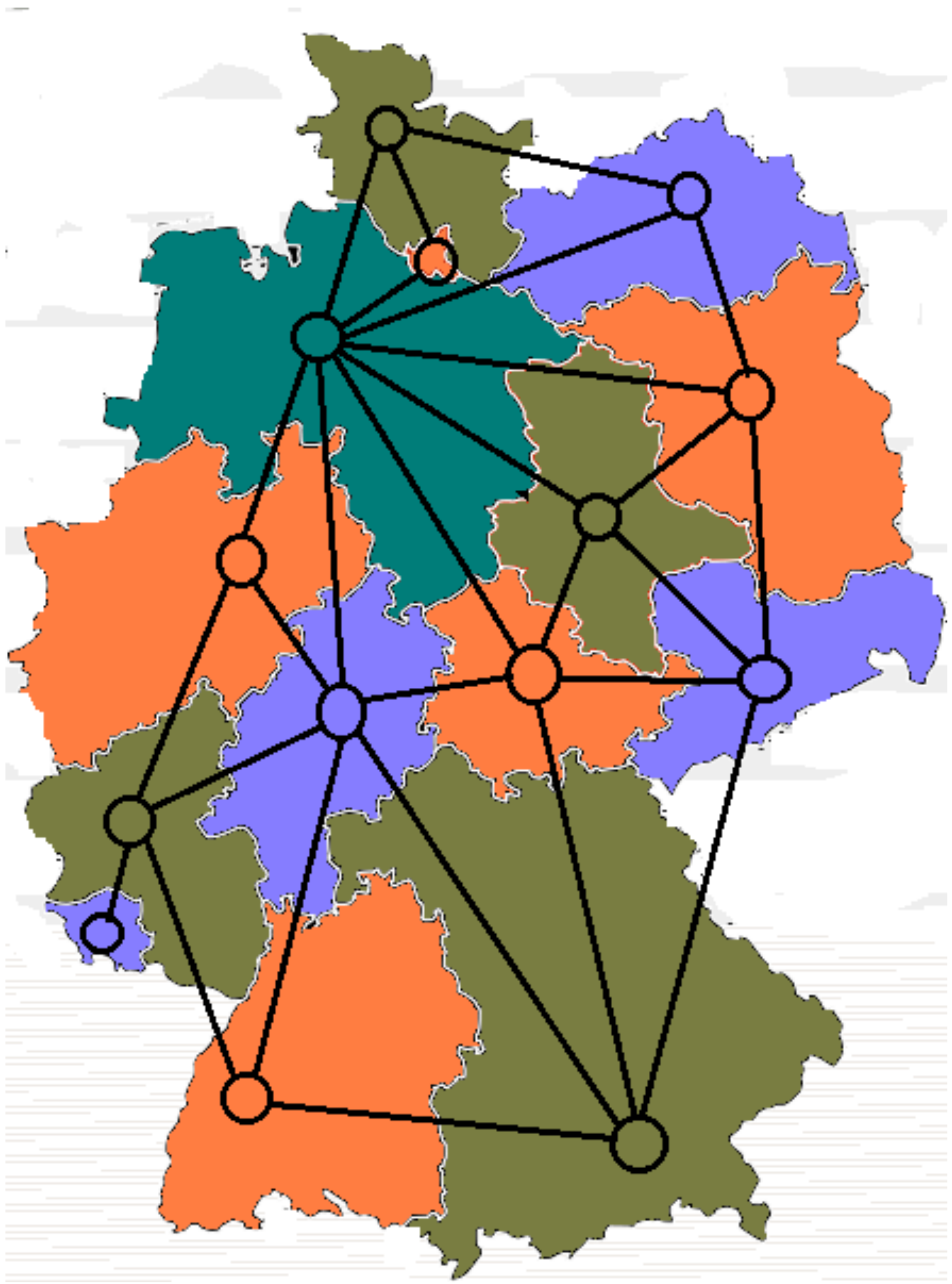
- If we look at a map we notice that two thing can complicate matters: Islands and lakes.
- More generally we see that non-connected countries will give us problems.
- It can be shown that if we allow non-connected countries we can find maps where the four-colour theorem is false.



A more exact formulation

- We require that the countries must be simply connected and have boundaries that are sufficiently simple.
- Furthermore, two countries meeting just in one point are not to be considered as neighbours.
- These complications make it natural to study the *dual* graph-form: Every planar graph can be coulored with four colours (in the normal node-colouring sense).

The dual graph

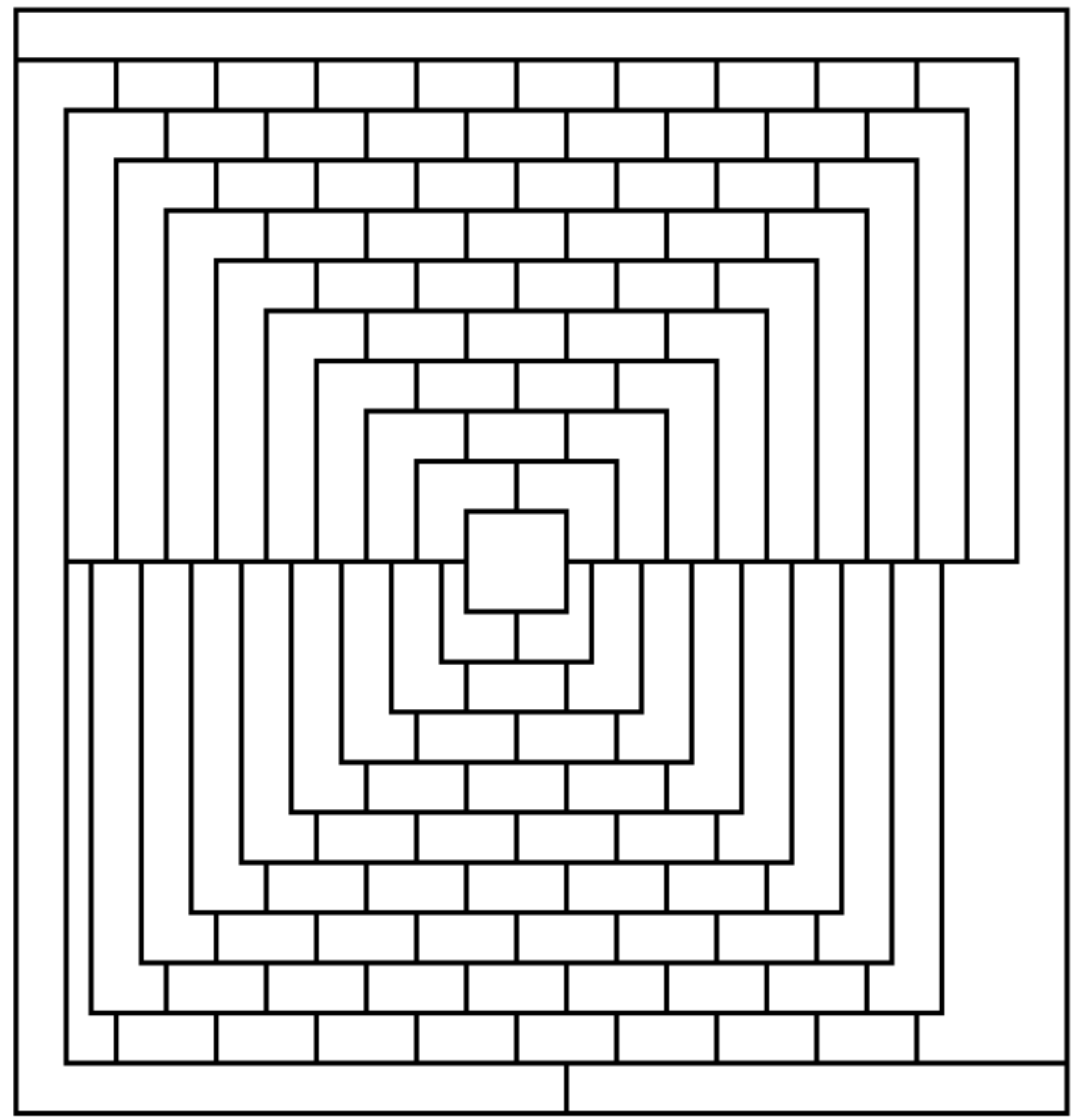


Method comments 1

- It is important to get an exact formulation of the problem as soon as possible.
- A problem can often be expressed in different forms. Even if the forms are equivalent, one of them can be easier to work with than the other.

True or false?

- When we face a conjecture we have to guess if it is true or not.
- If we think it is true we try to prove it.
- If we think it is false we try to find a counter-example.



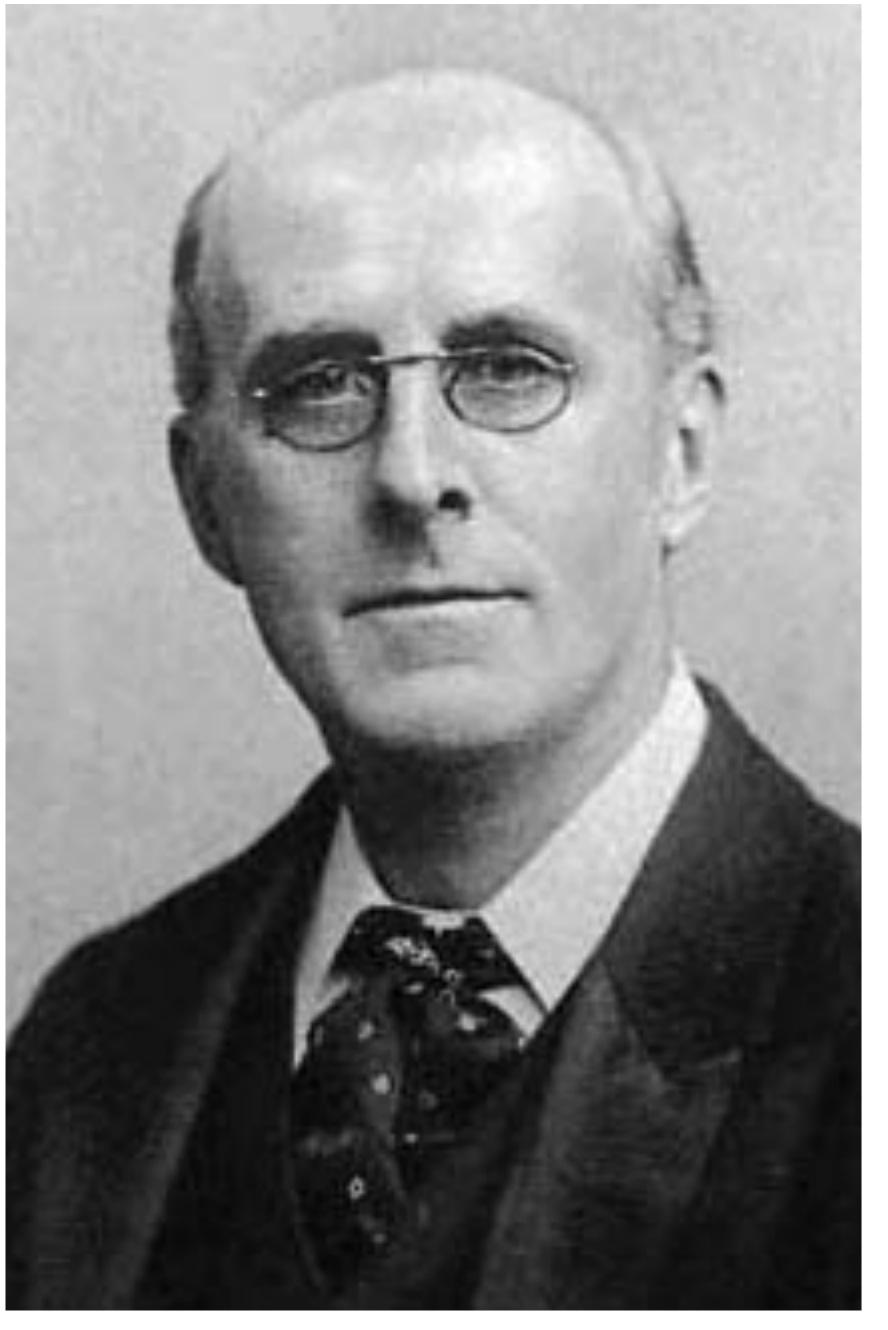
A counterexample?

A proof?

- Most people believed that the FC-Theorem was true. But how can we prove it?
- One attempt is to try to find an algorithm which actually colours any map with no more than four colours.
- But then we have to prove that the algorithm always manage to do this.
- We could try to find some more complicated existence-proof of a four-colouring.
- We could use mathematical induction.

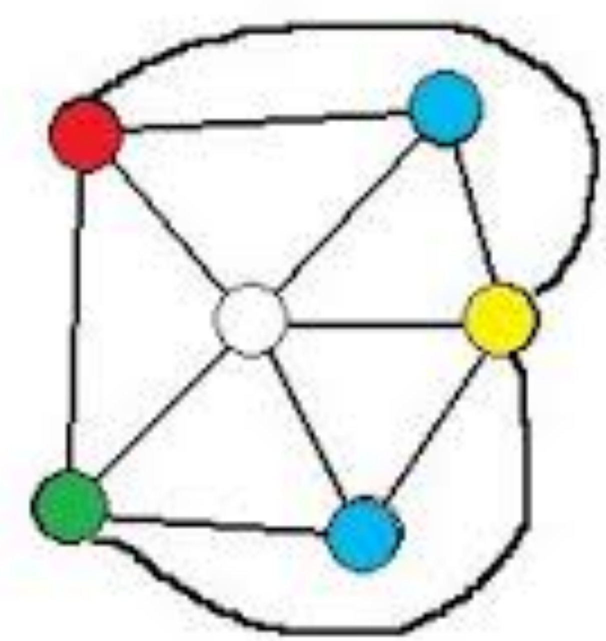
Kempe's "proof"

- In 1879 Sir Alfred Kempe managed to "prove" the FC-Theorem.
- He had a very good idea which used induction.
- He observed that all maps must contain at least one country surrounded with no more than five countries.



Details

- In the dual form we must have at least one node with degree no more than five.
- Remove the node and colour the rest of the graph with four colours.
- If, necessary, re-colour the graph so that no more than three colours are used around the start-node.
- Kempe "showed" that this can always be done.
- So then we can colour our graph with four colours!



Not so!

- In fact, the re-colouring which Kempe described does not work.
- This error was undiscovered for ten years!
- The error was then spotted by Heawood.

Method comments 2

- If a proof is erroneous, it means that there is a counterexample.
- Counterexamples come in two forms:
- Global counterexample - An example which shows that the statement in the theorem is false.
- Local counterexample - An example which shows that a step in the proof is incorrect.
- Kempe's proof fell due to a local counterexample (of course).

Algorithms

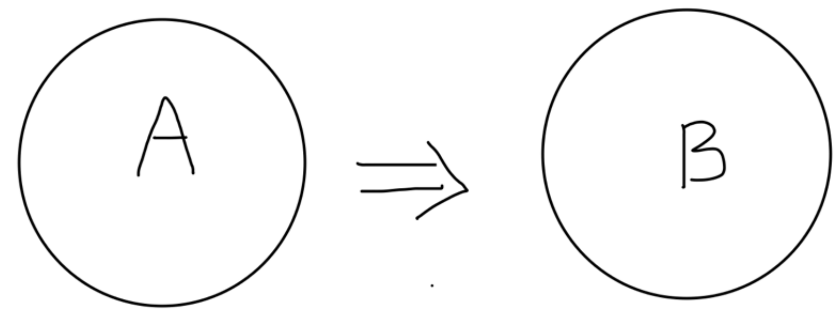
- We can apply the same reasoning to the correctness of algorithms.
- An algorithm takes an input and is supposed to deliver an output of a certain kind.
- An FC-algorithm take a plane graph as input and outputs a FC.
- We can speak of two kinds of counterexamples against the correctness of the algorithm:
- Global counterexample - An example which gives output on the wrong form.
- Local counterexample - An example which makes a certain step in the algorithm impossible to perform.

Method comments 3

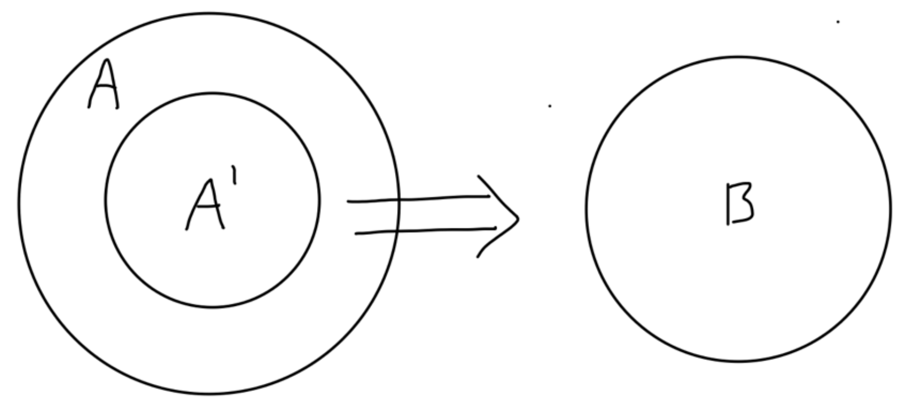
- Let us assume that we have a theorem of the form A => B. (For instance, A: A graph is plane B: The graph can be coloured with four colours.)
- We can *weaken* the theorem by replacing A or B with other statements. The weaker theorem can perhaps be proved.
- 1. Assume A' => A. Then A' => B is a *weakened* form of the theorem.
- 2. Assume B => B'. Then A => B' is a *weakened* form of the theorem.

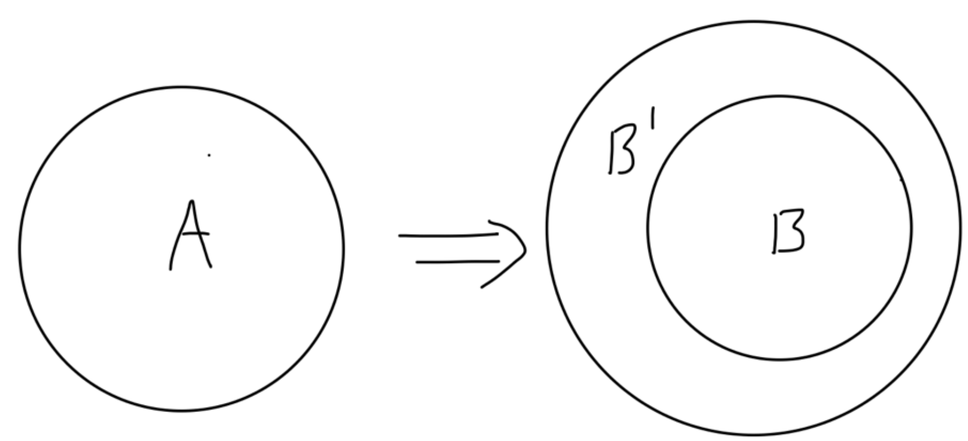
Weakening

The original theorem: A => B



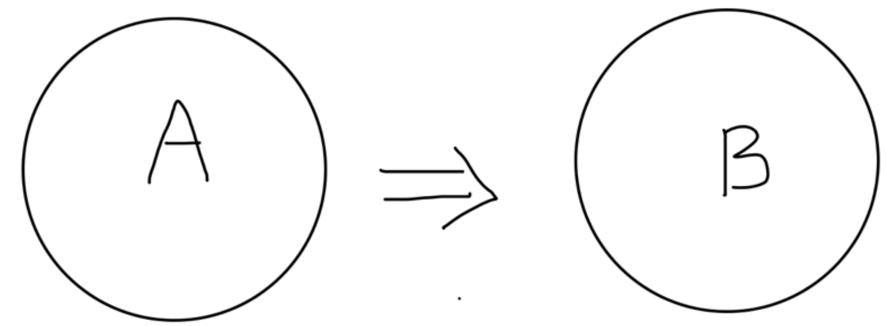
Weaker forms:



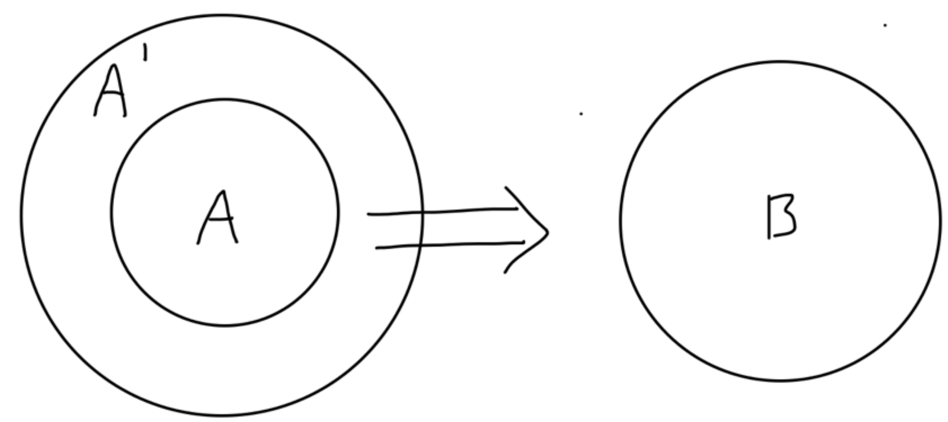


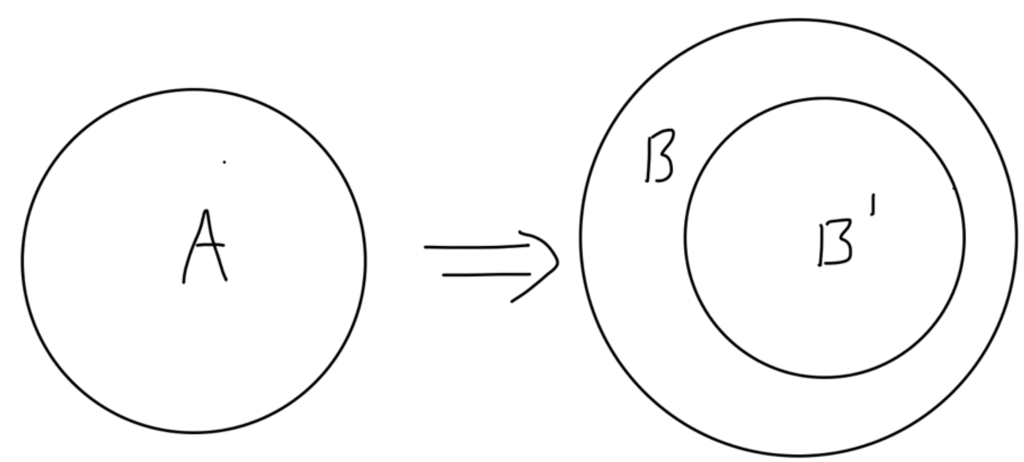
Strengthening

The original theorem: A => B



Stronger forms:



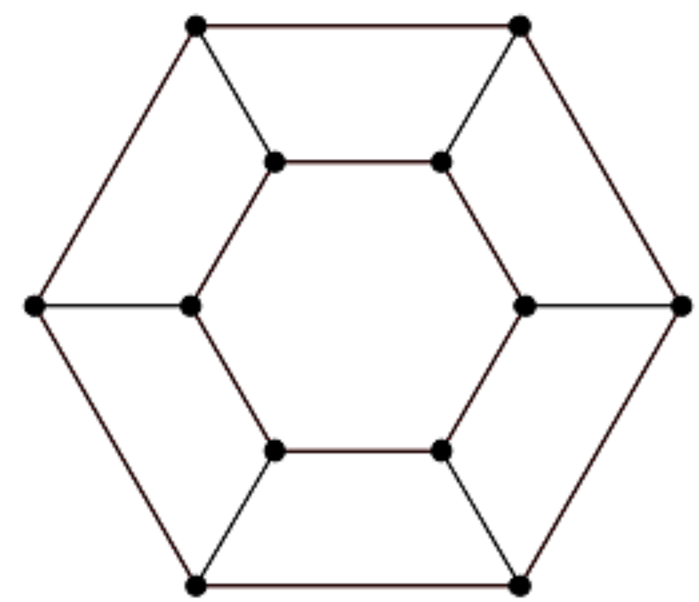


The Five-colour Theorem

- In 1890 Heawood used Kempe's technique and proved that every plane graph can be coloured with no more than five colours.
- It is obviously a weakening of the FC-Theorem.
- Heawood's proof shows that an erroneous proof (Kempe's) can still be useful.

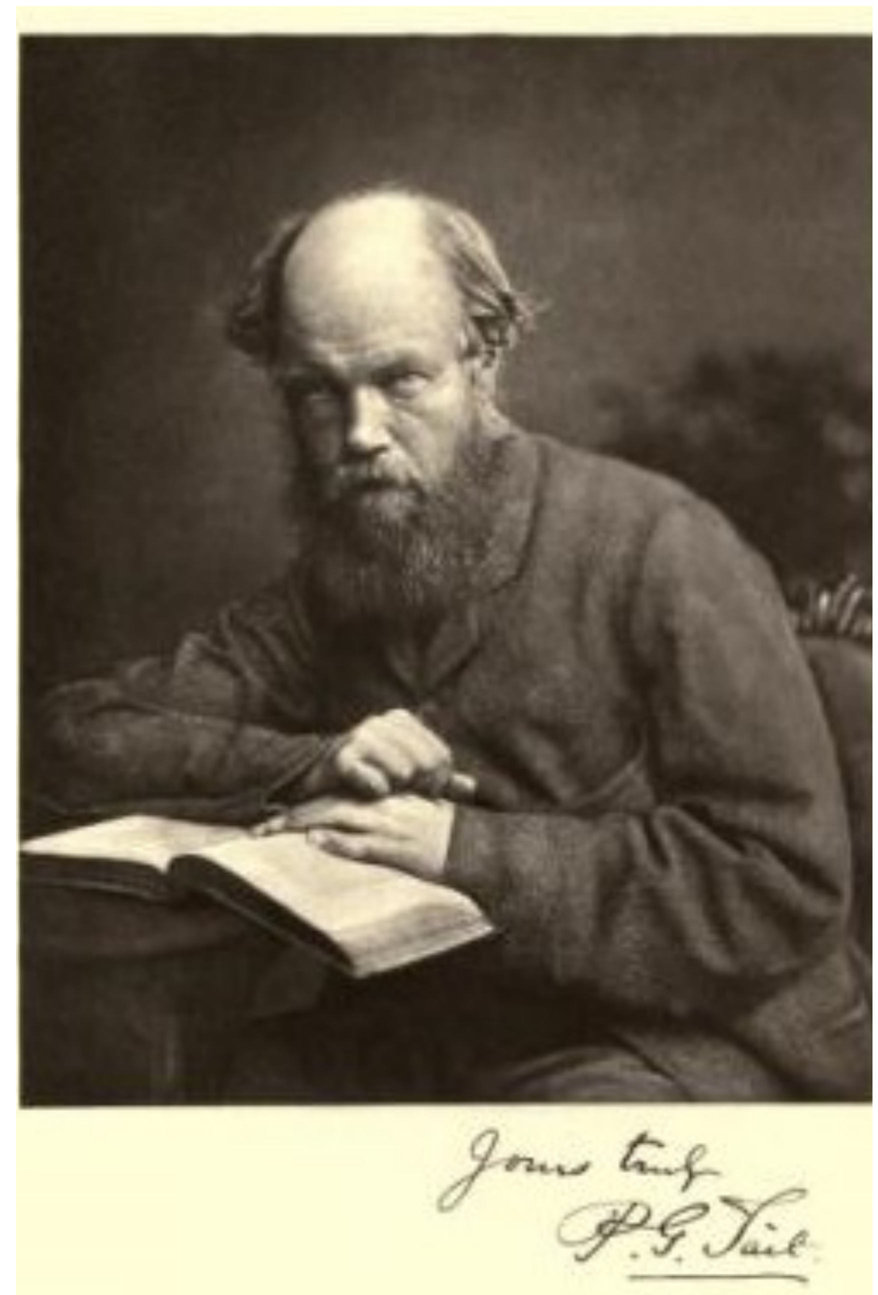
Another weakening

- Even before Kempe's proof it was known that it is enough to prove the FC-Theorem for *cubic* maps.
- Cubic maps - Maps where all nodes have degree three.



A reduction

- Tait managed to show that if we can show that every cubic map has a Hamiltonian Cycle, then the FC-Theorem must be true.
- But it turned out that there are (global) counterexamples to this statement, i.e. the existence of Hamiltonian Cycles.



A new idea: Edge-colourings

- Given a graph we can colour its edges. We say that a colouring is correct if any edges with a common node is coloured with different colours.
- Vizing's theorem: If N is the minimal number of colours needed to colour the graph G and D is the maximal node-degree in G, then N is either D or D+1.
- Tait showed that the FC-Theorem is true if and only if every plane bridgeless cubic graph can be edge-coloured with three colours.

Method comments 4

- We can speak about different problems. Informally we can say that Problem 2 is weaker than Problem 1 if a solution to Problem 1 would give us a solution to Problem 2.
- In the same way is Problem 1 stronger than Problem 2.
- And if a solution to any of the problems would give us a solution to the other one, we say that the problems are equivalent.

A comparison with Complexity Theory

- In complexity theory we have the notation ≤ where Problem 2 ≤ Problem 1 means that there is a polynomial time reduction from Problem 2 to Problem 1.
- In our more general discussion we do not have a *formal* definition of reductions in this sense.

What we have seen this far

- The problem of proving FCT for maps is equivalent to proving FCT to graphs.
- Heawood solved the weaker problem of proving that every plane graph can be 5-coloured.
- It was shown that FCT can be reduced to the (apparently weaker) problem of proving that every plane cubic map is three-colourable.
- Tait showed that FCT could be reduced to the problem of proving that every plane cubic graph has a Hamiltonian Cycle.
- Tait showed that FCT is equivalent to the problem of proving that every plane cubic graph can be edge-three-coloured.

Turning to harder problems

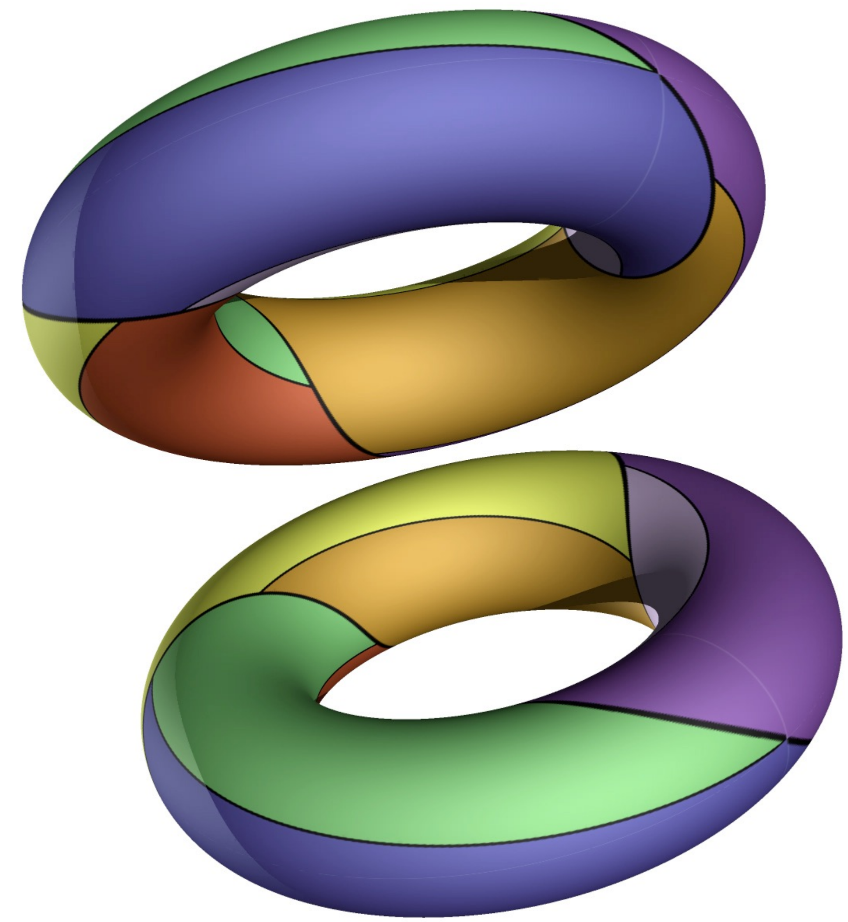
- It turned out that the FCT remained unproved despite all these promising approaches.
- What one could do then is to try to solve a harder problem.

Chromatic polynomials

- The mathematician Birkhoff tried to solve an apparently harder problem. He wanted to decide in how many ways an arbitrary graph G can be coloured with x colours.
- It turns out that the answer is a polynomial P(G,x), a so called *chromatic polynomial.*
- Birkhoff tried to show that for all planar graphs G we have $P(G,4) > 0$. But he didn't succeed.

Other types of maps

- Instead of plane maps we can consider maps on other bodies.
- For instance, on a torus it is quite easy to show that seven colours always suffice but not six colours.
- In fact, we can show variants of the FCT for all bodies except for spheres (which are equivalent to planes).



Method comments 5

- We have seen several promising attempts to prove the FCT. Eventually, none of them gave a proof.
- Nevertheless we see that trying to solve a problem can lead to other interesting problems and solutions to them.

The proof of the Four-colour Theorem

- The path towards the proof of the FCT starts with a return to Kempe's failed proof from 1879. The proof uses ideas that Kempe had.
- The proof uses induction over the size of the graph. Then we observe that a planar graph must have a set of *unavoidable* subgraphs.
- Then we prove that the subgraphs are *reducable*. This means that if the rest of the graph can be four-coloured, then this colouring can be extended to the subgraph with some minor changes to the original colouring.
- Kempe found the a simple unavoidable subgraph in form of a node with degree at most five. But he failed to prove that the subgraph is reducable (it is not).
- Appel and Haken had the idea that they should try to find more complicated unavoidable subgraphs.

A computer proof

- Appel and Haken managed to find a set of 1936 *together unavoidable* subgraph. (That means that in any planar graph at least one of the subgraphs must occur.)
- But in order to prove that the subgraphs were *reducable* they had to rely on a computer program to find the re-colouring strategies.
- The proof became much debated and criticized. It opened for a discussion of what a proof really is or should be.

Method comments 6

- So eventually the original idea by Kempe was triumphant.
- But in 1890 there was probably no easy way to see this.
- It was when all other strategies had failed that the return to the original idea seemed attractive.
- So sometimes a failed proof can be resurrected.