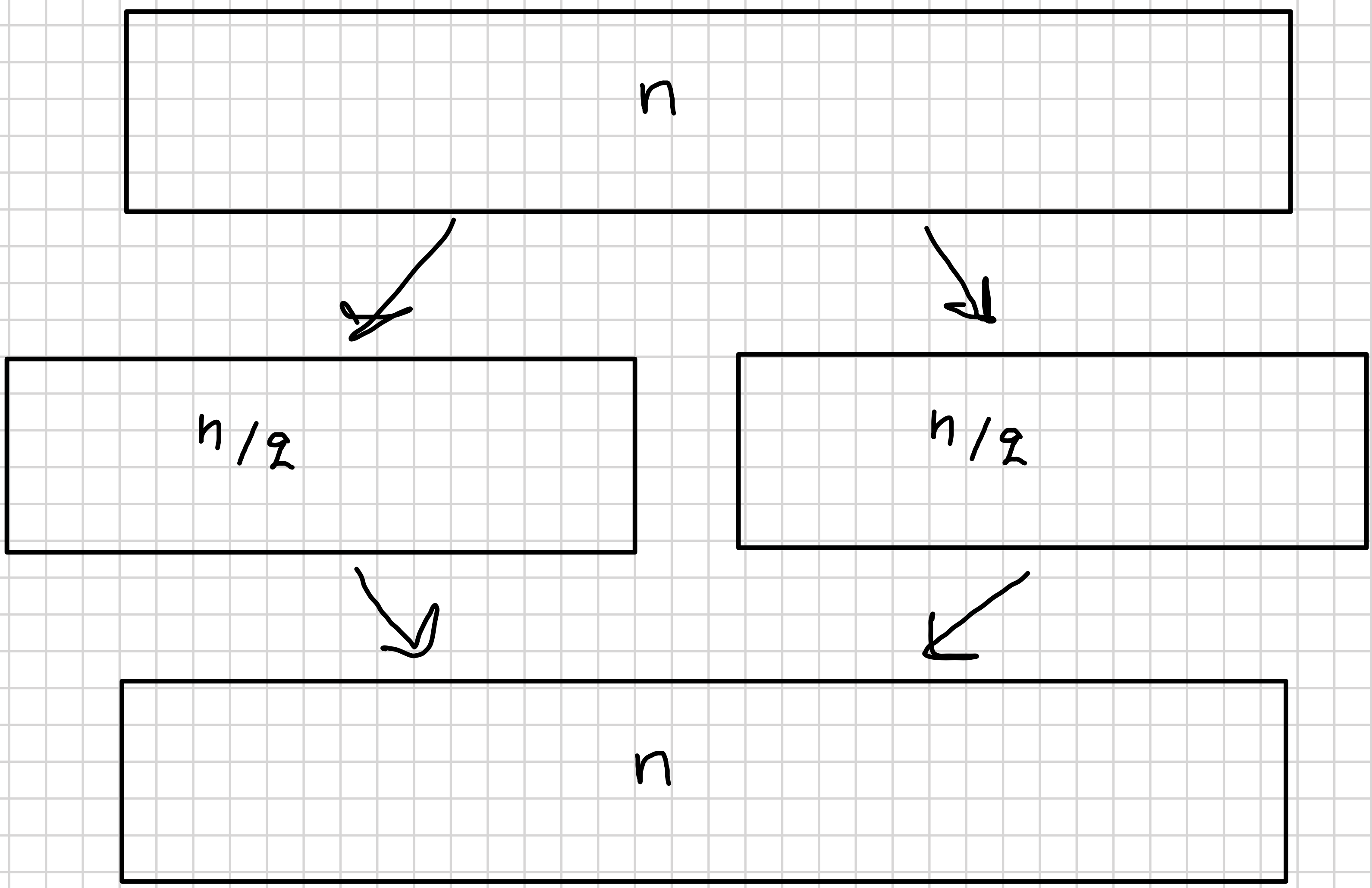
Divide and Conquer algorithms

Another general method for constructing algorithms is given by the Divide and Conquer strategy. We assume that we have a problem with input that can be split into parts in a natural way.



Let $T(n)$ be the time-complexity for solving a problem of size n (using our algorithm). Then we have $T(n) = T(n/2) + T(n/2) + f(n)$
where $f(n)$ is the time for "making the split" and "putting the parts together.
This will be useful only if $f(n)$ is sufficiently small.

Mergesort

A famous example is Mergesort. Here we split a list of numbers into two parts, sort them separately, and merge the two lists.

$n/2$ $\qquad$ $n/2$

How do we merge?

The question is how we merge two already sorted lists and what the complexity $f(n)$ is?
We can use the following algorithm:

```
Merge[a[1, ..., p], b[1,...,q]]
    If a = ∅
        Return b
    End if
    If b = ∅
        Return a
    End if
    If a[1] ≤ b[1]
        Return a[1] . Merge[a[2,...,p],b[1,...,q]]
    End if
    Return b[1] . Merge[a[1,...,p],b[2,...,q]]
```

The complexity is $O(n)$.

The main Mergesort algorithm is:

MergeSort

MergeSort($v[i..j]$)
(1) **if** $i < j$
(2) $m \leftarrow \left\lfloor \frac{i+j}{2} \right\rfloor$
(3) MergeSort($v[i..m]$)
(4) MergeSort($v[m+1..j]$)
(5) $v[i..j] =$ Merge($v[i..m], v[m+1..j]$)

Let $T(N)$ be the time it takes to sort N numbers. then

$$T(N) = \begin{cases} O(1) & N = 1 \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + \Theta(N) & \text{else} \end{cases}$$

since Merge $\Theta(N)$ when input is arrays of length N.

But how do we decide the complexity? We are given a recursion equation. The following theorem often gives the solution:

Master Theorem

Theorem *If $a \geq 1$, $b > 1$ and $d > 0$ the equation*

$$T(1) = d$$
$$T(n) = aT(n/b) + f(n)$$

has the solution

- $T(n) = \Theta(n^{\log_b a})$ *if* $f(n) = O(n^{\log_b a - \epsilon})$ *for some* $\epsilon > 0$

- $T(n) = \Theta(n^{\log_b a} \log n)$ *if* $f(n) = \Theta(n^{\log_b a})$

- $T(n) = O(f(n))$ *if* $f(n) = \Omega(n^{\log_b a + \epsilon})$ *for some* $\epsilon > 0$ *and* $af(n/b) \leq cf(n)$ *for some* $c < 1$ *for* n *large enough.*

When applied on Mergesort this theorem gives $\Theta(N \log N)$.

A special case of MT

If we assume that $f(n) = \Theta(n^d)$ for some integer d, we get a simpler formula. Let us first set $k = \log_b a$.

$$T(n) = \begin{cases} \Theta(n^k) & k > d \\ \Theta(n^k \log n) & k = d \\ \Theta(n^d) & k < d \end{cases}$$

It can be interesting to look at the special case $a = b$ $(k = 1)$

$$T(n) = \begin{cases} \Theta(n) & 1 > d \\ \Theta(n \log n) & 1 = d \\ \Theta(n^d) & 1 < d \end{cases}$$

And we can also look at $a = 1$, $b = 2$ $(k = 0)$

$$T(n) = \begin{cases} \Theta(\log n) & 0 = d \\ \Theta(n^d) & 0 < d \end{cases}$$

Let's look at some more advanced examples.

Multiplication of large numbers

We want to compute $x \cdot y$ for binary numbers x och y

$$x = \underbrace{x_{n-1} \cdots x_{n/2}}_{a} \underbrace{x_{n/2-1} \cdots x_1 x_0}_{b} = 2^{n/2} a + b$$

$$y = \underbrace{y_{n-1} \cdots y_{n/2}}_{c} \underbrace{y_{n/2-1} \cdots y_1 y_0}_{d} = 2^{n/2} c + d$$

For $n = 2^k$ we can split the product:

Mult(x, y)
(1) **if** $length(x) = 1$
(2) **return** $x \cdot y$
(3) **else**
(4) $[a, b] \leftarrow x$
(5) $[c, d] \leftarrow y$
(6) $prod \leftarrow 2^n Mult(a, c) + Mult(b, d)$
$+2^{n/2}(Mult(a, d) + Mult(b, c))$
(7) **return** $prod$

Time-complexity: $T(n) = 4T(n/2) + \Theta(n)$, $T(1) = \Theta(1)$ which gives us $T(n) = \Theta(n^2)$.

Here is a way of doing it that really uses D and C:

Karatsuba's algorithm

We use $(a+b)(c+d) = ac + bd + (ad + bc)$.
We can remove one of the four products:

```
Mult(x, y)
    if length(x) = 1
        return x · y
    else
        [a, b] ← x
        [c, d] ← y
        ac ← Mult(a, c)
        bd ← Mult(b, d)
        abcd ← Mult(a + b, c + d)
        return 2^n · ac + bd +
        2^{n/2}(abcd − ac − bd)
```

We get $T(n) = 3T(n/2) + \Theta(n)$, $T(1) = \Theta(1)$ with the solution $T(n) = \Theta(n^{\log_2 3}) \in O(n^{1.59})$. .

Here is an algorithm that fails to use D and C in a creative way.

Matrix multiplication

When we multiply $n \times n$-matrices we can use matrix blocks:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

by using the formulas

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

we get 8 products and

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ 8T(n/2) + \Theta(n^2) & n > 1 \end{cases}$$

which gives us $T(n) = \Theta(n^3)$.

But this is D and C:

.

Strassen's algorithm

If we instead use the more complicated formulas

$$
\begin{aligned}
M_1 &= (A_{12} - A_{22})(B_{21} + B_{22}) \\
M_2 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_3 &= (A_{11} - A_{21})(B_{11} + B_{12}) \\
M_4 &= (A_{11} + A_{12})B_{22} \\
M_5 &= A_{11}(B_{12} - B_{22}) \\
M_6 &= A_{22}(B_{21} - B_{11}) \\
M_7 &= (A_{21} + A_{22})B_{11}
\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= M_1 + M_2 - M_4 + M_6 \\
C_{12} &= M_4 + M_5 \\
C_{21} &= M_6 + M_7 \\
C_{22} &= M_2 - M_3 + M_5 - M_7
\end{aligned}
$$

we reduce the number of products to 7 which gives us $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$.

An advanced application of D and C is the Fast Fourier Transform (FFT). We start by describing what the Discrete Fourier Transform (DFT) is:

Discrete Fourier Transform

We transform a polynomial $A(x) = \sum_{j=0}^{n-1} a_j x^j$. Essentially we do it by computing it's values for the complex unity roots $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$ where $\omega_n = e^{2\pi i/n}$.

$$DFT_n(\langle a_0, \ldots, a_{n-1} \rangle) = \langle y_0, \ldots, y_{n-1} \rangle$$

where

$$y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j e^{2\pi ijk/n}.$$

The n coefficients gives us n "frequencies". Compare with the continuous transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} dx$$

This simplest way of computing this transform has complexity $O(n^2)$. The FFT is a more efficient way of doing it.

FFT: An efficient way of computing DFT

We have $y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j e^{2\pi ijk/n}$. We separate odd and even degrees in A:

For $k < n/2$ We have

$$
\begin{aligned}
A^{[0]}(\omega_n^{2k}) &= \sum_{j=0}^{n/2-1} a_{2j} e^{4\pi ijk/n} \\
&= \sum_{j=0}^{n/2-1} a_{2j} \omega_{n/2}^{jk} \\
&= DFT_{n/2}(\langle a_0, a_2, \ldots, a_{n-2} \rangle)_k
\end{aligned}
$$

where $DFT_n(\langle a_0, \ldots, a_{n-1} \rangle)_k$ is the k:th element of the transform.

In the same way, for $k < n/2$,

$$A^{[1]}(\omega_n^{2k}) = DFT_{n/2}(\langle a_1, a_3, \ldots, a_{n-1}\rangle)_k$$

For $k \geq n/2$ we can easily see that

$$A^{[0]}(\omega_n^{2k}) = DFT_{n/2}(\langle a_0, a_2, \ldots, a_{n-2}\rangle)_{k-n/2}$$
$$A^{[1]}(\omega_n^{2k}) = DFT_{n/2}(\langle a_1, a_3, \ldots, a_{n-1}\rangle)_{k-n/2}$$
$$\omega_n^k = -\omega_n^{k-n/2}$$

In order to decide $DFT_n(\langle a_0, \ldots, a_{n-1}\rangle)$ we use $DFT_{n/2}(\langle a_0, a_2, \ldots, a_{n-2}\rangle)$ and $DFT_{n/2}(\langle a_1, a_3, \ldots$ and combine values.

FFT is a Divide Conquer algorithm — the base case is $DFT_1(\langle a_0\rangle) = \langle a_0\rangle$.

Algorithm for computing FFT

We assume that n is a power of 2.

$$
\begin{array}{l}
DFT_n(\langle a_0, a_1, \ldots a_{n-1} \rangle) \\
\quad \textbf{if } n = 1 \\
\quad\quad \textbf{return } \langle a_0 \rangle \\
\quad \omega_n \leftarrow e^{2\pi i/n} \\
\quad \omega \leftarrow 1 \\
\quad y^{[0]} \leftarrow DFT_{n/2}(\langle a_0, a_2, \ldots, a_{n-2} \rangle) \\
\quad y^{[1]} \leftarrow DFT_{n/2}(\langle a_1, a_3, \ldots, a_{n-1} \rangle) \\
\quad \textbf{for } k = 0 \textbf{ to } n/2 - 1 \\
\quad\quad y_k \leftarrow y_k^{[0]} + \omega y_k^{[1]} \\
\quad\quad y_{k+n/2} \leftarrow y_k^{[0]} - \omega y_k^{[1]} \\
\quad\quad \omega \leftarrow \omega \cdot \omega_n \\
\quad \textbf{return } \langle y_0, y_1, \ldots, y_{n-1} \rangle
\end{array}
$$

The time-complexity $T(n)$ is given by

$$
T(n) = \begin{cases} O(1) & n = 1 \\ 2T(n/2) + \Theta(n) & n > 1 \end{cases}
$$

with solution $T(n) = \Theta(n \log n)$.

Inverse to DFT

The relation $y = DFT_n(a)$ can be written in matrix form

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \omega_n^0 & \omega_n^0 & \cdots & \omega_n^0 \\ \omega_n^0 & \omega_n^1 & \cdots & \omega_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^0 & \omega_n^{n-1} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

To get the inverse transformation $a = DFT_n^{-1}(y)$ we invert the matrix It can be shown that

$$DFT_n^{-1}(\langle y_0, y_1, \ldots, y_{n-1} \rangle) = \langle a_0, a_1, \ldots, a_{n-1} \rangle$$

$$a_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k \omega_n^{-jk}$$

so the FFT-algorithm can also be used to compute DFT^{-1}.

Polynomial multiplication using FFT

We want to compute $C(x) = \sum_{j=0}^{2n-2} c_i x^i = A(x)B(x)$ when $A(x)$ and $B(x)$ are polynomials of degree $n-1$. Since $C(x)$ has $2n-1$ coefficients we will look at $A(x)$ and $B(x)$ as polynomials of degree $2n-1$ as well.

Algorithm:

$$\langle y_0, \dots y_{2n-1} \rangle \leftarrow DFT_{2n}(\langle a_0, \dots, a_{n-1}, 0, \dots, 0 \rangle)$$
$$\langle z_0, \dots z_{2n-1} \rangle \leftarrow DFT_{2n}(\langle b_0, \dots, b_{n-1}, 0, \dots, 0 \rangle)$$
$$\langle c_0, \dots c_{2n-1} \rangle \leftarrow DFT_{2n}^{-1}(\langle y_0 z_0, \dots, y_{2n-1} z_{2n-1} \rangle)$$

(We assume that n is a power of two.)

We have to do compute three DFT vectors of size $2n$ and compute $2n$ products in the transform plane. That gives us the complexity $\Theta(n \log n)$.