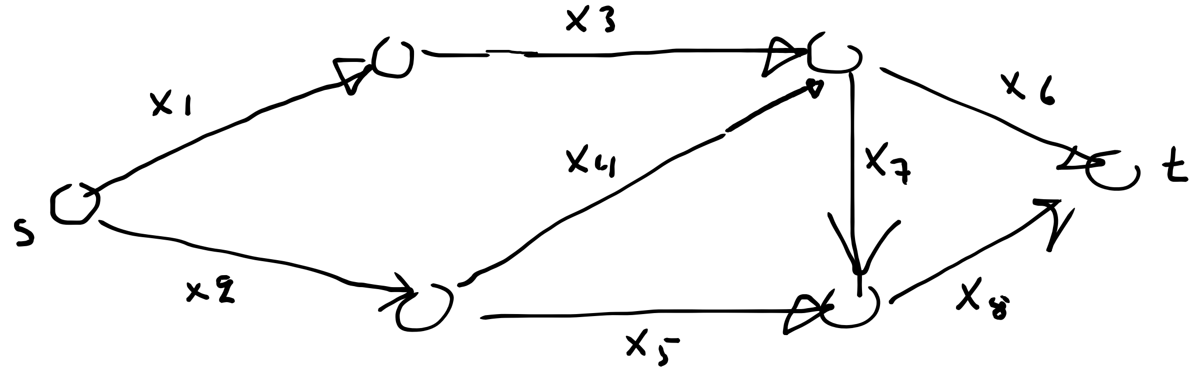
Linear Programming

Linear Programming are about what first might seem as a very special type of problem. These problem can be solved with the Simplex algorithm. A concept called duality will be of great importance. A reason for studying these problems is that a lot of other problems can be reduced to linear programming problems. A first example will be flow problems.

Ex:



Try to maximize x1+x2

when
x1 = x3
x4+x5 = x2
x6+x7 = x3+x4
x8=x5+x7

and
0.< xi
xi < ci

for all i

This is a special form of linear programming problems.

The Flow Problem as LP problem

We let x_e be the flow on edge e. We have the constraints $0 \leq x_e \leq c(e)$ for all e. For each node x except s and t we have

$$\sum_{e \in In(x)} x_e = \sum_{e \in Out(x)} x_e$$

We set

$$v = \sum_{e \in Out(s)} x_e$$

The flow problem can be written as
Maximize v
when

$$\begin{cases} v = \sum_{e \in Out(s)} x_e & \\ \sum_{e \in In(x)} x_e = \sum_{e \in Out(x)} x_e & \text{for all x except s,t} \\ 0 \leq x_e \leq c(e) & \text{for all edges} \end{cases}$$

A transport problem

The company Carla produces milk in 4 different plants. The milk is delivered to 5 customers. Carla has to consider three things:

1. The capacities of the plants.
2. The demands of the customers.
3. The costs of the transports between plants and customers.

Let us call the plants F1, F2, F3, F4.

Capacity:

F1	F2	F3	F4
30	40	30	40

(The numbers represent 1000 liters.)

Let us call the customers K1, K2, K3, K4, K5.

Demand:

K1	K2	K3	K4	K5
20	30	15	25	20

(The numbers represent 1000 liters.)

Transport costs:

	K1	K2	K3	K4	K4
F1	$2,80$	$2,55$	$3,25$	$4,30$	$4,35$
F2	$4,30$	$3,15$	$2,55$	$3,30$	$3,50$
F3	$3,00$	$3,30$	$2,90$	$4,30$	$3,40$
F4	$5,20$	$4,45$	$3,50$	$3,75$	$2,45$

Goal:

Decide how the ”flow” to the customers should be so that

1. The customers are satisfied.
2. The cost are minimal.

Mathematical model:

Use variables x_{ij} for the flow from plant i to customer j.

What demands do we have?

1. Capacities

Ex: For plant 1 we should have
$x_{11}+x_{12}+x_{13}+x_{14}+x_{15} \leq 30000$

2. Demand

Ex: For customer 1 we should have
$x_{11}+x_{21}+ x_{31}+x_{41} = 20000$

Cost:

z = 2,80 x_{11} + 2,55 x_{12} + ... + 2,45 x_{45}

We use the following definitions:

Let c_{ij} be the cost for transport from plant i to customer j.

Let s_i be the capacity for plant i.

Let d_j be the demand of customer j.

The problem can now be written as

Minimize $\sum_{i=1}^{4} \sum_{j=1}^{5} c_{\mathrm{ij}} x_{\mathrm{ij}}$

when
$$\begin{array}{c} \sum_{j=1}^{5} x_{\mathrm{ij}} \leqslant s_i \;\; i=1,2,3,4 \\ \sum_{i=1}^{4} x_{\mathrm{ij}} = d_j \;\; j=1,2,3,4,5 \\ x_{\mathrm{ij}} \geqslant 0 \end{array}$$

Linear Programming

A linear programming problem on generic form is

Minimize $\sum_{j=1}^{n} c_j x_j$

when
$\sum_{j=1}^{n} a_{\mathrm{ij}} x_j \leqslant b_i \;\; i=1,2,...,m$
$x_{j \geqslant 0}$

If we have a problem that is not on general form we can rewrite it on general form. We show how it can be done by looking at some examples
Example:
Minimize
$$x_1 + 2x_2 - x_3$$
when
$$\begin{cases} x_1 + x_3 = 1 \\ x_2 - x_3 \geq 3 \end{cases}$$
We can change minimization to maximization by changing sign on the function.
Inequalities "in the wrong direction"can be turned right by a sign change.
Equalities can be turned into inequalities by using two using two inequalities for each equality.
In our problem we get
Maximize
$$-x_1 - 2x_2 + x_3$$
when
$$\begin{cases} x_1 + x_3 \leq 1 \\ -x_1 - x_3 \leq -1 \\ x_3 - x_2 \leq -3 \end{cases}$$

Example: A company called Fajo

The company Fajo makes bandy sticks and hockey sticks.
There are two steps in the production: Sawing and glueing.

There are times needed for the two steps

	Sawing	Gluing	
Hockey	7	16	(minutes)
Bandy	10	12	

The firm has capacity for 3600 minutes of sawing and 5400 minutes of gluing per week.

The sticks can be sold for
Hockey: 125 kr but has a production cost of 105 kr.
Bandy: 115 kr but has a production cost of 97 kr.

Let x_1 be the number of produced hockey x_2 the number of produced bandy sticks.

We get the following problem

Maximize z = $20x_1 + 18x_2$

when
$x_1 + 10x_2 \leq 3600$
$16x_1 + 12x_2 \leq 5400$
$x_1, x_2 \geq 0$

Obs: We can transform the problem to general form if we say that we want to minimize -20x_1-18x_2

There is a famous algorithm called the Simplex Algorithm that solves these problems. We will describe this algorithm without going to much into details.

The Simplex Method

Preparation: We transform the problem to so called standard form

Standard form: We have equalities instead of inequalities.

Ex:

Minimize z = $3x_1$ + $5x_2$ - x_3

when

$x_1 - x_2 + 2x_3 = 5$
$x_1 + 2x_2 + 4x_3 = 12$
$x_1, x_2, x_3 \geqslant 0$

We get equalities by introducing Slack Variables.

Ex: Let us assume that we have the inequality $x_1 + 3x_2 \leqslant 10$

We set $x_3 = 10 - (x_1 + 3x_2)$

x_3 is a new slack variable.

We get the equality $x_1 + 3x_2 + x_3 = 10$

Standard firm can be described with matrices:

Minimize z = $\sum_{j=1}^{n} c_j x_j$

when
$\sum_{j=1}^{n} a_{\mathrm{ij}} x_j = b_i$ i = 1,... ,m
$x_j \geqslant 0$ j = 1, ... ,n

We can use matrix notation

Minimize $\bar{c}^T \bar{x}$

when
$A\bar{x} = \bar{b}$
$\bar{x} \geqslant \bar{0}$

The Fajo problem put on standard form will be:

$7x_1 + 10x_2 \leqslant 3600$ reduces to $7x_1 + 10x_2 + x_3 = 3600$

$16x_1 + 12x_2 \leqslant 5400$ reduces to $16x_1 + 12x_2 + x_4 = 5400$

We get

Maximize z = $20x_1 + 18x_2$

when
$7x_1 + 10x_2 + x_3 = 3600$
$16x_1 + 12x_2 + x_4 = 5400$
$x_1, x_2, x_3, x_4 \geqslant 0$

In Matrix form it looks like:

$$A = \begin{pmatrix} 7 & 10 & 1 & 0 \\ 16 & 2 & 0 & 1 \end{pmatrix}$$

and

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} -20 \\ -18 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 3600 \\ 5400 \end{pmatrix}$$

Minimize (-20, -18, 0, 0) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

when

$$\begin{pmatrix} 7 & 10 & 1 & 0 \\ 16 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leqslant \begin{pmatrix} 3600 \\ 5400 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

How to find a solution

Maximize z = $20x_1+ 18x_2$.
When

$7x_1+ 10x_2+ x_3 = 3600$
$16x_1 + 12x_2 + x_4= 5400$

How do we find the best solution?

One possibility is $x_3= x_4= 0$

$7x_1+ 10x_2 = 3600$
$16x_1 + 12x_2 = 5400$

If we solve the system we get $x_1\approx 142$ $x_2 \approx 260$

It gives us z $\approx$ 7520

But instead, we can put $x_2 = x_4= 0$

We get the equations

$7x_1+ x_3 = 3600$
$16x_1 = 5400$

They give us $x_1 \approx 337$ $x_3\approx 1237$

Then z $\approx$ 2362.

Are there more solutions?

Basic solution:

Let us assume that we have n variables and m equations. We also assume that all equations are linearly independent. We us say that we have set n-m of the variables to 0.
Then the other m variables have unique values. This gives us a basic solution .

Feasible basic solution:

If all variables are $\geqslant 0$ we have a feasible basic solution.

The solution to a LP-problem is always a feasible basic solution (FBS).

But which FBS?

Method:

Variables which are 0 (at a certain stage) are called non-basic variables. The other variables are called basic variables.

We test different FBS:s by changing the basic variables one at a time.

Ex: Minimize z = $2x_1 + x_2$

when $3x_1 + x_2 = 10$
$x_1, x_2 \geqslant 0$

Set $x_1 = 0$.

Then $x_2 = 10$ and z = 10.

We now change basic variables so that $x_2 = 0$.

Then $x_1 \approx 3{,}33$

we get z $\approx$ 6,67.

So we have found a better solution.

How do you know if you have found the best solution?

Ex: Fajo

$x_1 = 142$ $x_2 = 260$ $z = 7520$

Is that the best solution?

We can write

$x_3 = 3600 - 7x_1 - 10x_2$
$x_4 = 5400 - 16x_1 - 12x_2$

$x_1 = 0{,}158x_3 - 0{,}132x_4 + 142{,}1$
$x_2 = -0{,}2x_3 + 0{,}092x_4 + 260{,}5$

That gives us z = $20x_1 + 18x_2 = 20(0{,}15x_3 - 0{,}13x_4 + 142{,}1) + 18(-0{,}21x_3 + 0{,}09x_4 + 260{,}5) =$
$7520 - 0{,}62x_3 - 0{,}98x_4$

Now we see that we would gain nothing by increasing x_3 or x_4.
We see that any change from this solution must end in a worse solution.

General description of the Simplex Method

Let's say that we have a maximization problem and a FBS with basic variables y_1, y_2, ... , y_m and non-basic variables v_1, v_2 , ... , v_{n-m}.

This means that $v_1 = v_2 = ... = v_{n-m} = 0$

We can then write y_1, y_2, ... , y_m as functions of v_1, v_2 , ... , v_{n-m}

$y_1 = f_1 (v_1, ..., v_{n-m})$ $y_2 = f_2 (v_1, ..., v_{n-m})$
...
In the same way we can write z as

z $= c_1 v_1 + c_2 v_2 + ... + c_{n-m} v_{n-m} + z_0$

If all c_i are < 0 we must have an optimal solution.

If any $c_i > 0$, say $c_1 > 0$, we can increase z by increasing v_1. But then the values of the y:s must change. How much do they change?

We can increase v_1 until $f_k(v_1, v_2, ...) = 0$ for some k. Then v_1 will be a new basic variable and y_k will be a new non-basic variable.
We go on like this until all $c_i \leq 0$. Then we have found the optimal solution.

If we have a minimization problem we must try to increase variables with $c_i < 0$. When all $c_i \geq 0$ we have a solution.

Ex:

Minimize z $= 2x_1 + 2x_2 + x_3$

when
$x_1 + x_2 + x_3 = 5$
$x_1 - x_2 + 2x_3 = 8$
$x_1, x_2, x_3 \geqslant 0$

One FBS is $x_2 = 0$ (non-basic variable).

We get

$x_1 + x_3 = 5 - x_2$
$x_1 + 2x_3 = 8 + x_2$

$x_1 = 2 - 3x_2$
$x_3 = 3 + 2x_2$

$z = 2(2 - 3x_2) + 2x_2 + (3 + 2x_2) = 7 - 2x_2$

We can increase x_2. But how much?

x_1 and x_3 must be $\geqslant 0$.

$x_1 = 2 - 3x_2$

This means $x_2 \leqslant \frac{2}{3}$

$x_3 = 3 + 2x_2$

This gives us no bound on x_2.

So $x_2 = \frac{2}{3}$ and $x_1 = 0$.

$x_3 = \frac{13}{3}$

We now write x_2, x_3 as functions of x_1.

$x_2 = \frac{2}{3} - \frac{x_1}{3}$

$x_3 = 3 + 2x_2 = 3 + 2(\frac{2}{3} - \frac{x_1}{3}) = \frac{13}{3} - \frac{2x_1}{3}$

$z = 7 - 2x_2 = 7 - 2(\frac{2}{3} - \frac{x_1}{3}) = \frac{17}{3} + \frac{2x_1}{3}$

Since we gain nothing by increasing x_1, we are done.

This is however far from the full story. There is a problem called degeneracy that can occur. This happens when when we have no $c_i > 0$ and some $c_i = 0$ (if we assume that we have a minimization problem). In that case we will have to chose some i with $c_i = 0$. Then there is a chance that we could get into an infinite cycle. In practice, there are several ways to avoid this. Another problem is how to find a starting point for the algorithm. It turns out that we can use a modified variant of the simplex algorithm to solve this problem.

Actually, in worst case, the Simplex Algorithm is not a polynomial time algorithm. In practice, however, it is always considered efficient enough to be used.

Dual Problems

For each LP problem we can give a so called dual problem.

Ex: We have the problem

Maximize $5x_1 + 2x_2$

when
$x_1 + x_2 \leqslant 10$
$2x_1 + 3x_2 \leqslant 20$
$x_1 , x_2 \geqslant 0$

The dual problem is

Minimize $10v_1 + 20v_2$

when
$v_1 + 2v_2 \geqslant 5$
$v_1 + 3v_2 \geqslant 2$
$v_1, v_2 \geqslant 0$

How do we define the dual problem?

We write the problem on the form

Maximize $\bar{c}^T \bar{x}$

when
$A\bar{x} \leqslant \bar{b}$
$\bar{x} \geqslant \bar{0}$

The dual problem is

Minimize $\bar{b}^T \bar{v}$

when
$A^T \bar{v} \geqslant \bar{c}$
$\bar{v} \geqslant \bar{0}$

The Duality Theorem

Let P_1 and P_2 be two dual problems. If one of the problems has a unique solution with value M, then the other problem also has a unique solution with value M. If we solve one of the problems we also get a solution to the other.
Ex: Fajo again

We want to

Maximize $20x_1 + 18x_2$

when
$7x_1 + 10x_2 \leqslant 3600$
$16x_1 + 12x_2 \leqslant 5400$
$x_1, x_2 \geqslant 0$

The corresponding dual problem is

Minimize $3600v_1 + 5400v_2$

when
$7v_1 + 16v_2 \geqslant 20$
$10v_1 + 12v_2 \geqslant 18$
$v_1, v_2 \geqslant 0$

Both problems have the same value as solution.

But what does the dual problem mean?

Let us assume that Fajo want to rent out its production facilities. What rent would the marketbe willing to pay? We can suppose that the market will pay v_1 kr/minute for sawing and v_2 kr/minute for gluing.
What prices v_1 and v_2 should the market set?

The market will want to minimize $3600v_1 + 5400v_2$

The market must also consider the following requirements: Fajo must want to rent out. This means that Fajo must make at least as much money as it would if it run the production itself.
A hockey stick can be sold with a profit of 20 kr. It will take 7 minutes of sawing and 10 minutes of gluing to make it. When Fajo rents out it would get $7v_1 + 16v_2$ kr . This number must be at least 20.

$7v_1 + 16v_2 \geqslant 20$

In the same way we get
$10v_1 + 12v_2 \geqslant 18$.

This gives us

Minimize $3600v_1 + 5400v_2$

when
$7v_1 + 16v_2 \geqslant 20$
$10v_1 + 12v_2 \geqslant 18$
$v_1, v_2 \geqslant 0$

About solutions to LP problems

If we try to solve a LP problem three cases can occur.

1. The problem has a unique solution.

2. The inequalities defining the problem cannot be satisfied.

Ex: Minimize $x_1 + x_2$

when
$x_1 + 3x_2 \leqslant -2$
$x_1 + 3x_2 \geqslant 1$
$x_1, x_2 \geqslant 0$

Then, of course, there is no solution to the problem.

3. The value can be arbitrarily large/small.

Ex: Maximize $x_1 - x_2$

when
$x_1 + x_2 \geqslant 10$
$x_1, x_2 \geqslant 0$
The problem is that $x_1 - x_2$ can be arbitrarily large. There is no solution.

Part of what the Duality Theorem tells us is that if one of two dual problems is of type 1, the other one must be of type 1 as well.

Reduction of a problem to a LP problem

Example: Find the shortest path $s \to t$ in a weighted graph G.

Maximize d_t

when

$$\begin{cases} d_u \leq d_v + w(u,v) & \text{for all edges}(u,v) \\ d_s = 0 \end{cases}$$

Examples of dual problems

The flow problem can be put on dual form:
The vector $\bar{y}$ contains $|V| + |E|$ numbers. They are g_i for each node v_i and γ_j for each edge e_j.
Minimize

$$\sum_j \gamma_j c_j$$

when

$$\begin{cases} g_i - g_j + \gamma_k \geq 0 & \text{om} \quad e_k = (v_i, v_j) \\ g_n - g_1 \geq 1, \quad \gamma_j \geq 0 & \text{for all} \quad j \end{cases}$$

The solution to this problem generates a minimal cut $(S, V-S)$ and an assignment of values $g_i = 0$ if $v_i \in S$, $g_i = 0$ otherwise. $\gamma_j = 1$ if e_j goes from S to $V - S$, $\gamma_j = 0$ otherwise.

A translation of the shortest path problem to dual form gives us:

Maximize

$$\sum_e x_e w(e)$$

when

$$\begin{cases} 1 = \sum_{e \in Ut(s)} x_e \\ \sum_{e \in In(x)} x_e = \sum_{e \in Ut(x)} x_e & \text{for all x except s,t} \\ 0 \leq x_e \leq 1 & \text{for all edges} \end{cases}$$