Algorithms and Complexity
2016
Mästarprov 2: Complexity

Mästarprov 2 should be solved individually in written form and presented orally. No collaboration is allowed.

Written solutions should be handed in latest on **Monday, April 25th 17.00,** to Johan or Mladen in written or printed form (personally or physical mailbox). Be sure to save a copy of your solutions. Mästarprov 1 is a mandatory and rated part of the course. The test consists of four tasks. The test is roughly graded as follows: Two task correctly solved give an E. Three tasks correctly solved give a C and all tasks correctly solved give an A. You can read more about the grading criteria and the final grade on the course web page. The report should be written in English.

In all problems you should give an analysis of the time complexity of your algorithm and you should be able to argue for its correctness. When making reductions you are free to use any NP-Complete problems mentioned in the course book or in the lecture notes.

1.

We have a one person puzzle game played on an $m \times n$ board. On each of its mn positions lies either a blue chip, a red chip or nothing at all. You play by removing chips from the board. You win if you can remove chips so that each column contains only chips of a single color and each row contains at least one chip. Winning may or may not be possible, depending upon the initial configuration. We can now state the following problem: Given an initial configuration on an $m \times n$ board, decide if we can reach a winning position or not.Show that this problem is in NP and show that it is NP-Hard by reducing CNF-SAT to this problem.

2.

Let us assume that we want to send a questionary to the citizens in a city. Let us say that the population P is a set of persons and that we have disjoint groups $A_1, A_2, ..., A_t$ of people that interests us. These groups do not necessarily cover all of P. The city is divided into areas with zip codes $z_1 z_2, ..., z_n$. These codes give a partitioning of P. We also have numbers m_{ij} such that m_{ij} is the number of persons in A_j living in the area with code z_i. We cannot expect all people to answer the questionary. We have numbers r_j such that r_j is the probability that a randomly selected person in A_j will answer the questionary. Let us now consider this problem: Given two vectors $(l_1, l_2, ..., l_t)$ and $(h_1, h_2, ..., h_t)$, can we find a set of zip codes such that the expected values $E_1, E_2, ..., E_t$ of numbers of people answering in each group in all the chosen zip code areas are is such that $l_j \leq E_j \leq h_j$ for $j = 1, 2, ..., t$?

Show that this problem is in NP and show that it is NP-Hard by reducing SUBSET SUM to this problem.

3.

We can imagine that we have a computer network N with computers $C_1, C_2, ..., C_n$ forming a graph. The edges E represent pairs of computers that can send messages to each other. We now have an authority A that wants to infiltrate the network. A has identified m possible dangerous groups $G_1, G_2, ..., G_m$ of users. A identify the users with their computers, so the groups G_i can be regarded as subsets of $V(N)$. The groups are not necessarily disjoint. What A plan to do is to infiltrate the groups by hijacking some of the computers. It means that A can monitor all conversation going trough the computers. If A can monitor at least one computer in a group, that group is considered as monitored. The groups all have an importance index I_j $j = 1, , ..., m$. These are integers in $\{1, 2, ..., 10\}$. There is a cost w_i for monitoring each computer C_i. A has a cost budget, the sum of the costs w_i is not allowed to exceed W. Given that constraint A wants to find a way to monitor as many important groups as possible. A works with a very simple success rate: Let S be the sum of the importance indices of all the groups monitored. Then S is the success rate.

A now faces the problem of, given a constraint W, choosing computers to monitor such the cost does not exceed W and such that the success rate S is as large as possible. This is an optimization problem. Formulate a corresponding decision problem and show that it is NP-complete.

4.

Let us now assume that the authority A from problem 3 has come over a machine M that tells us if it is possible to monitor a network or not. More precisely, the machine takes as input X an encoding $(V(N), E(N))$, a list $\bar{G}$ of the groups and $\bar{I}$ of their importance, a list $\bar{w}$ of the monitoring costs and a budget constraint W. Given this input, M decides the best possible success rate S in time $T(|X|)$. A does not know what T is exactly but thinks that it is rather efficient. But what A wants is not just the best possible S, it also wants the actual best choice of computers to monitor. Construct such an algorithm that takes X as input and outputs an optimal choice of computers and makes no more than a polynomial number of calls on M. This means that the complexity should be $O(p(|X|)T(|X|))$ where p is some polynomial. Compute this complexity (that is, p) and prove the correctness of your algorithm.