DD2422 Exercises 1
with solutions

Hugo Cornelius, Per Rosengren

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Part I
Problems

1. Given homogeneous coordinates of a point, $x = (x_1, x_2, x_3)^T$ and a line $l = (l_1, l_2, l_3)^T$ in a plane, the equation of the line can be written as $l^T x = 0$. Show that this notation can be used to represent:

(a) intersection x between two lines l and l_0 as $x = l \times l_0$,
(b) line l defined by two points x and x_0 as $l = x \times x_0$. where "$\times$" denotes cross product.

2. Projective transformations between two planes (that also include perspective transformation between a plane in a world and the image plane) can be represented by the following expressions:

$$y = Ax$$

where $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ represent homogeneous coordinates in these two planes respectively. Here, A is non-singular 3×3-matrix. Estimate how are the following geometric entities transform under this specific transformation:

(i) a line $l^T x = 0$, where l represents a vector of length 3,
(ii) an ellipse $x^T C x = 0$, where C represents a positive definite 3×3-matrix.

(ii)

3. A grey level image defined as $f : \Omega \to [0, z_{max}]$ has a histogram of the following form:

$$p(z) = \frac{\pi}{2z_{max}} \sin(\frac{\pi}{2} \frac{z}{z_{max}})$$

Define a montonic grey level transformation function $T : [0, z_{max}] \to [-\frac{z_{max}}{2}, \frac{z_{max}}{2}]$ such that grey levels in the transformed image $g(x, y) = T(f(x, y))$ (where $(x, y) \in \Omega$) are uniformly distributed in teh intervall $[-\frac{z_{max}}{2}, \frac{z_{max}}{2}]$. For which $z \in [0, z_{max}]$ will the estimated transformation result in stretching of the grey level values?

4. Estimate the result of convolution $h_i * f$ with $i = 1, 2, 3$ where

$$f = (\ldots, 0, 1, 2, 3, 11, 4, 0, \ldots)$$

and

$$h1 = (1, 2, 4),$$
$$h2 = (+1, 0, -1)$$
$$h3 = (1, -2, 1).$$

5. Estimate Fourier transform of vector $(1, 2, 3, 5)$.

6. If the mirroring of a function is defined as $f_-(x) = f(-x)$, show that $h_- * f_- = (h * f)_-$.

7. Estimate Fourier tarnsform of a triangle shaped filter, that along each coordinate direction has the following shape:

$$f(x_k) = \begin{cases} 1 + x_k & \text{om } -1 < x_k < 0, \\ 1 - x_k & \text{om } 0 < x_k < 1, \\ 0 & \text{annars} \end{cases}$$

Draw how the Fourier transform looks like and explain what are the unwanted effects of such a filter when used on images.

Part II
Solutions

1 Homogeneous coordinates

1.1 Intersection of two lines

Homogeneous coordinates are defined as

$$\overline{x} = c \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{1}$$

The intersection $\overline{x}$ is defined by

$$\begin{cases} l_a^T \overline{x} = 0 \\ l_b^T \overline{x} = 0 \end{cases} \tag{2}$$

This gives

$$\begin{cases} l_{a1}cx & + & l_{a2}cy & = & -l_{a3}c \\ l_{a1}cx & + & l_{a2}cy & = & -l_{a3}c \end{cases} \tag{3}$$

dividing by c:

$$\begin{cases} l_{a1}x & + & l_{a2}y & = & -l_{a3} \\ l_{a1}x & + & l_{a2}y & = & -l_{a3} \end{cases} \tag{4}$$

Expressed as matrix multiplication

$$\begin{pmatrix} l_{a1} & l_{a2} \\ l_{b1} & l_{a2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -l_{a3} \\ -l_{b3} \end{pmatrix} \tag{5}$$

Using 2x2 Matrix inversion, Beta p. 94:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{l_{a1}l_{b2} - l_{a2}l_{b1}} \begin{pmatrix} l_{b2} & -l_{a2} \\ -l_{b1} & l_{a1} \end{pmatrix} \begin{pmatrix} -l_{a3} \\ -l_{b3} \end{pmatrix} \tag{6}$$

Which gives the result

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{l_{a1}l_{b2} - l_{a2}l_{b1}} \begin{pmatrix} l_{a2}l_{b3} - l_{a3}l_{b2} \\ l_{a3}l_{b1} - l_{a1}l_{b3} \end{pmatrix} \tag{7}$$

From the exercise we have

$$\overline{x} = l_a \times l_b \tag{8}$$

Using equation 1 we get

$$c \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} l_{a2}l_{b3} - l_{a3}l_{b2} \\ l_{a3}l_{b1} - l_{a1}l_{b3} \\ l_{a1}l_{b2} - l_{a2}l_{b1} \end{pmatrix} \tag{9}$$

which gives

$$c = l_{a1}l_{b2} - l_{a2}l_{b1} \tag{10}$$

Inserting equation 10 in 9 gives equation 7.

1.2 Line from two points

We have that

$$l = \overline{x}_a \times \overline{x}_b \tag{11}$$

and

$$l^T \overline{x} = 0 \tag{12}$$

for all points $\overline{x}$ on the line l. Equation 12 is true for all vectors l that are orthogonal to $\overline{x}$. Equation 11 gives a vector l that is orthogonal to both $\overline{x}_a$ and $\overline{x}_b$. That means that equation 12 is true for both $\overline{x} = \overline{x}_a$ and $\overline{x} = \overline{x}_b$.

2 Projective transformations

We have

$$y = Ax \tag{13}$$

and since A is non-singular

$$x = A^{-1}y \tag{14}$$

2.1 Line

The line in the first plane is

$$l_a^T x = 0 \tag{15}$$

Equation 14 gives

$$l_a^T A^{-1} y = 0 \tag{16}$$

Rearranging, and using the notation A^{-T} for the transpose of the inverse of A, gives

$$\left(A^{-T} l_a\right)^T y = 0 \tag{17}$$

Which gives the line in the second plane

$$l_b^T y = 0 \tag{18}$$

where

$$l_b = A^{-T} l_a \tag{19}$$

2.2 Ellipse

The same method can be applied to an ellipse. The ellipse in the first plane is

$$x^T C_a x = 0 \tag{20}$$

Equation 14 gives

$$y^T A^{-T} C_a A^{-1} y = 0 \tag{21}$$

Which gives the ellipse in the second plane

$$y^T C_b y = 0 \tag{22}$$

where

$$C_b = A^{-T} C_a A^{-1} \tag{23}$$

3 Histogram normalisation

We have

$$p(z) = \frac{\pi}{2z_{\mathrm{max}}} \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \tag{24}$$

The transfer function will be of the form

$$T(z) = c_0 + c_1 \int_0^z p(z)\,\mathrm{d}z \tag{25}$$

where

$$T(0) = -\frac{z_{\mathrm{max}}}{2} \qquad \text{and} \qquad T(z_{\mathrm{max}}) = \frac{z_{\mathrm{max}}}{2} \tag{26}$$

That gives

$$T(0) = c_0 + c_1 \int_0^0 p(z)\,\mathrm{d}z = c_0 \Leftrightarrow c_0 = -\frac{z_{\mathrm{max}}}{2} \tag{27}$$

and

$$\begin{aligned}
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \int_0^{z_{\mathrm{max}}} p(z)\,\mathrm{d}z && (28)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \frac{\pi}{2z_{\mathrm{max}}} \int_0^{z_{\mathrm{max}}} \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \mathrm{d}z && (29)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \frac{\pi}{2z_{\mathrm{max}}} \frac{-2z_{\mathrm{max}}}{\pi} \left[\cos\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right)\right]_0^{z_{\mathrm{max}}} && (30)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1(-1)(0-1) && (31)\\
&\Leftrightarrow \\
c_1 &= T(z_{\mathrm{max}}) + \frac{z_{\mathrm{max}}}{2} = z_{\mathrm{max}} && (32)
\end{aligned}$$

The transfer function is therefore

$$\begin{aligned}
T(z) &= -\frac{z_{\mathrm{max}}}{2} + z_{\mathrm{max}} \int_0^z p(z)\,\mathrm{d}z && (33)\\
&\Leftrightarrow \\
T(z) &= -\frac{z_{\mathrm{max}}}{2} + z_{\mathrm{max}} \frac{\pi}{2z_{\mathrm{max}}} \int_0^z \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \mathrm{d}z && (34)\\
&\Leftrightarrow \\
T(z) &= -\frac{z_{\mathrm{max}}}{2} + \frac{\pi}{2} \int_0^z \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \mathrm{d}z && (35)
\end{aligned}$$

Stretching of grey level values is defined by

$$
\begin{aligned}
\frac{\partial T}{\partial z} &> 1 && (36)\\
&\Leftrightarrow \\
z_{\max}\, p(z) &> 1 && (37)\\
&\Leftrightarrow \\
\frac{\pi}{2}\sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) &> 1 && (38)\\
&\Leftrightarrow \\
\sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) &> \frac{2}{\pi} && (39)\\
&\Leftrightarrow \\
\frac{\pi}{2}\frac{z}{z_{\max}} &> \arcsin\left(\frac{2}{\pi}\right) \quad \text{when} \quad 0 \le z \le z_{\max} && (40)\\
&\Leftrightarrow \\
z &> z_{\max}\frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\right) && (41)
\end{aligned}
$$

4 Convolution

$$
\begin{aligned}
F &= \{0, 1, 2, 3, 11, 4, 0\} && (42)\\
G_1 &= \{1, 2, 4\} && (43)\\
G_2 &= \{1, 0, -1\} && (44)\\
G_3 &= \{1, -2, 1\} && (45)
\end{aligned}
$$

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y \tag{46}$$

$$f(x) = \begin{cases} F(x) & \text{if } 1 \le x \le 7 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{47}$$

$$g(x) = \begin{cases} G(2+x) & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

$$f * g(x) = \begin{cases} F * G(x-1) & \text{if } 2 \le x \le 6 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{49}$$

$$F * G(x-1) = \begin{cases} \sum_{y=-1}^{1} F(x-y)G(2+y) & \text{if } 2 \le x \le 6 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{50}$$

$$
\begin{aligned}
F * G_1 &= \{4, 11, 25, 38, 52\} && (51)\\
F * G_2 &= \{2, 2, 9, 1, -11\} && (52)\\
F * G_3 &= \{0, 0, 7, -15, 3\} && (53)
\end{aligned}
$$

5 Discrete Fourier transform

Whe have

$$v(0\dots3) = 1, 2, 3, 5 \tag{54}$$

The discrete Fourier transform is

$$\hat{v}(m) = \frac{1}{N}\sum_{n=0}^{N-1} v(n)e^{-i\,\frac{n}{N}m\,2\pi} \tag{55}$$

This gives

$$\hat{v}(0) = \frac{1}{4}(1+2+3+5) \tag{56}$$
$$\hat{v}(1) = \frac{1}{4}(1(1)+2(-i)+3(-1)+5(i)) \tag{57}$$
$$\hat{v}(1) = \frac{1}{4}(1(1)+2(-1)+3(1)+5(-1)) \tag{58}$$
$$\hat{v}(1) = \frac{1}{4}(1(1)+2(+1)+3(-1)+5(-i)) \tag{59}$$
$$\Leftrightarrow$$
$$\hat{v} = \frac{1}{4}(11,\ 3i-2,\ -3,\ -2-3i) \tag{60}$$

6 Mirroring

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \tag{61}$$
$$f_- * g_-(x) = \int_{-\infty}^{\infty} f_-(x-y)g_-(y)dy \tag{62}$$
$$f_- * g_-(x) = \int_{-\infty}^{\infty} f(-x-(-y))g(-y)dy \tag{63}$$
$$f_- * g_-(x) = \int_{\infty}^{-\infty} f(-x-y^*)g(y^*)(-dy^*) \text{ where } y^* = -y \tag{64}$$
$$f_- * g_-(x) = \int_{-\infty}^{\infty} f(-x-y^*)g(y^*)dy^* = f * g(-x) \tag{65}$$
$$f_- * g_- = (f * g)_- \tag{66}$$

7 Continuous Fourier transform

We have in spatial space

$$f(x) = \begin{cases} 1+x & \text{if} \quad -1 < x < 0 \\ 1-x & \text{if} \quad 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

This function is the convolution of two box functions

$$f(x) = g * g(x) \tag{68}$$

where

$$g(x) = \theta(x + \frac{1}{2}) - \theta(x - \frac{1}{2}) = f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{69}$$

The Fourier transform of g is (see Beta p. 319, F50)

$$\hat{g}(w) = \frac{2}{w} \sin\left(\frac{1}{2}w\right) \tag{70}$$

That together with equation 68 and the Fourier transform of a convolution (Beta p. 317, F13) gives

$$\hat{f}(w) = \hat{g}^2(w) = \frac{4}{w^2} \sin^2\left(\frac{w}{2}\right) \tag{71}$$

This is what the function looks like: