Exercises 2 - suggested solutions

Babak Rasolzadeh

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8.1
Ett en-dimensionellt klassificeringsproblem med avseende på storheten z omfattar tre klasser A, B och C med a priori-sannolikheterna p_A, p_B och p_C och frekvensfunktionerna $f_A(x)$, $f_B(x)$ och $fB(x)$, där

$p_A = 1/2, f_A(x) = 1/8$ då $x \in [-4, 4]$
$p_B = 1/3, f_B(x) = 3(1 - x^2)/4$ då $x \in [-1, 1]$
$p_C = 1/6, f_C(x) = x/8$ då $x \in [0, 4]$

Beräkna optimala klassificeringsgränser och beslutsregler för tilldelning av klasstillhörighet enligt ovan angiven statistik. Förklara de generella principer du baserar din lösning på och rita en illustrativ figur som förtydligar dina beräkningar och resultatet.

Answer: In general in a classification problem with a number of classes C_k, with *a priori* probabilities p_{C_k} and pdf's $f_{C_k}(x)$, the optimal decision rule for a given outcome x is to choose the class C_{opt} that maximizes the expression:

$$C_{opt} = \arg\max_{C_k} p_{C_k} \cdot f_{C_k}(x)$$

Here we set:
$g_A(x) = p_A \cdot f_A(x) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ when $x \in [-4, 4]$
$g_B(x) = p_B \cdot f_B(x) = \frac{1}{3} \cdot \frac{3(1-x^2)}{4} = \frac{1-x^2}{4}$ when $x \in [-1, 1]$
$g_C(x) = p_C \cdot f_C(x) = \frac{1}{6} \cdot \frac{x}{8} = \frac{x}{48}$ when $x \in [0, 4]$

In order to find the classification borders we need to solve the following system of inequalities:

$$\begin{cases} g_A(x) > g_B(x) & (1) \\ g_A(x) > g_C(x) & (2) \\ g_B(x) > g_C(x) & (3) \end{cases}$$

so that $x \in C_k$, where

$$C_k = \begin{cases} A & if & (1) \wedge (2) \\ B & if & (3) \wedge \neg(1) \\ C & if & \neg(2) \wedge \neg(3) \end{cases}$$

Now, let us solve this system. Equation (1) yields:

$$\frac{1}{16} > \frac{1-x^2}{4} \Leftrightarrow x^2 > \frac{3}{4} \Rightarrow \begin{cases} x > \frac{\sqrt{3}}{2} \vee x < -\frac{\sqrt{3}}{2} \end{cases}$$

Likewise, equation (2) yields:

$$\frac{1}{16} > \frac{x}{48} \Leftrightarrow x < 3$$

Before we calculate the solution to equation (3), let us plot the three functions $g_{C_k}(x)$.See Figure 1

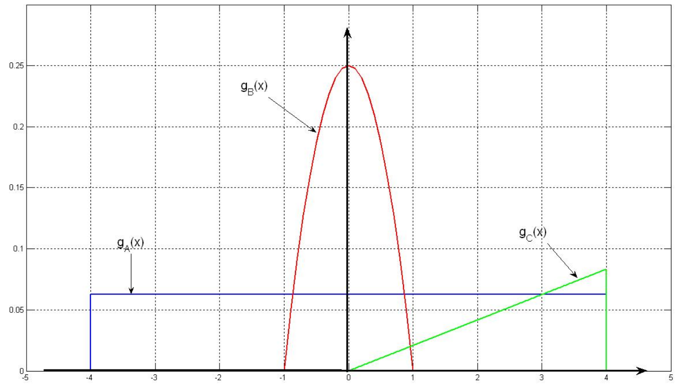

Figure 1: Illustration of the classification problem

We see that the solution to equation (3) is obsolete, since it's equivalent with the negated solution to equation (1) and the solution to equation (2). (Reason is that $3 > \sqrt{3}/2$). In other words, the classification boundaries are:

$$C_k = \begin{cases} A & if & -4 < x < -\frac{\sqrt{3}}{2} \quad \vee \quad \frac{\sqrt{3}}{2} < x < 3 \\ B & if & -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2} \\ C & if & 3 < x < 4 \end{cases}$$

8.2

För ett klassificeringsproblem med två klasser C_A och C_B, låt a priori-sannolikheterna vara $p_A = 3/4$ och $p_B = 1/4$. Låt vidare sannolikhetsfördelningarna svara mot normalfördelningar

$$p(\bar{z}|C_k) = \frac{1}{2\pi|det\Sigma_k|^{1/2}} e^{-(\bar{z}-m_k)^T\Sigma_k^{-1}(\bar{z}-m_k)/2}$$

med

$$m_A = m_B = 0,$$

och

$$\Sigma_A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \Sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Beräkna beslutsgränserna för detta klassificeringsproblem!

Answer: The decision boundary is given by the ”equal probability solution”: $Pr(A|\bar{x}) = Pr(B|\bar{x})$. Now multiply each side with $Pr(\bar{x})$, giving $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(\bar{x}) \cdot Pr(B|\bar{x})$. However, according to Bayes' rule, $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(A) \cdot Pr(\bar{x}|A)$ which gives us tbar the decision boundary is given by $Pr(A) \cdot Pr(\bar{x}|A) = Pr(B) \cdot Pr(\bar{x}|B)$. Thus we need to solve the equation

$$p_A \cdot p(\bar{x}|A) = p_B \cdot p(\bar{x}|B)$$

for $\bar{x} = [xy]$. Since we have the expression for $p(\bar{x}|C_k)$ and p_{C_k}, this is trivial. We will solve the equation as an inequality for the case $\bar{x} \in A$, i.e. $p_A \cdot p(\bar{x}|A) > p_B \cdot p(\bar{x}|B)$ (this is to have correct boundary conditions):

$$\frac{p_A}{2\pi \left|\det \Sigma_A\right|^{1/2}} e^{\frac{-(\bar{x}-m_A)^T \Sigma_A^{-1} (\bar{x}-m_A)}{2}} > \frac{p_B}{2\pi \left|\det \Sigma_B\right|^{1/2}} e^{\frac{-(\bar{x}-m_B)^T \Sigma_B^{-1} (\bar{x}-m_B)}{2}}$$

$$\Leftrightarrow \frac{3}{4} \frac{2}{2\pi\sqrt{4}} e^{\frac{-\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}{2}} > \frac{1}{4} \frac{2}{2\pi\sqrt{4}} e^{\frac{-\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}{2}}$$

$$\Leftarrow \ln 3 - \begin{bmatrix} x/8 & y/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > - \begin{bmatrix} x/4 & y/8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Leftrightarrow \ln 3 - \left(\frac{x^2}{8} + \frac{y^2}{2}\right) > - \left(\frac{x^2}{2} + \frac{y^2}{8}\right)$$

$$\Leftrightarrow x^2 - y^2 + \frac{8}{3} \ln 3 > 0$$

Now define $d(\bar{x}) \equiv d(x, y) \equiv x^2 - y^2 + \frac{8}{3} \ln 3$, then our classification rule becomes:

$$C_k = \begin{cases} A & if \quad d(\bar{x}) > 0 \\ B & if \quad d(\bar{x}) < 0 \end{cases}$$

Note that $d(x, y) = 0$ defines a hyperbolic curve, see Figure 2

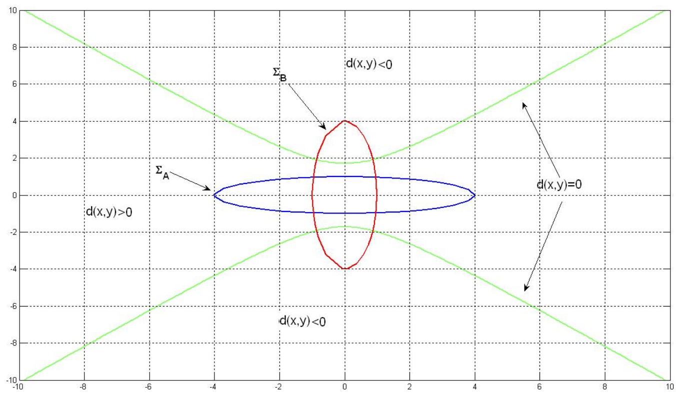

Figure 2: Illustration of the 2D classification problem

10.5

Beskriv hur man kan beräkna den räta linje som minimerar summan av de vinkelräta avstånden till en punktmängd. Genomför detta förfarande för följande bildpunkter:

$$p_1 = (-4, -2), p_2 = (-2, -1), p_3 = (0, 0), p_4 = (1, 1), p_5 = (3, 2)$$

Answer: Note that this is different from a least squares estimation where it is the *vertical* distances that are minimized. See Figure 3.

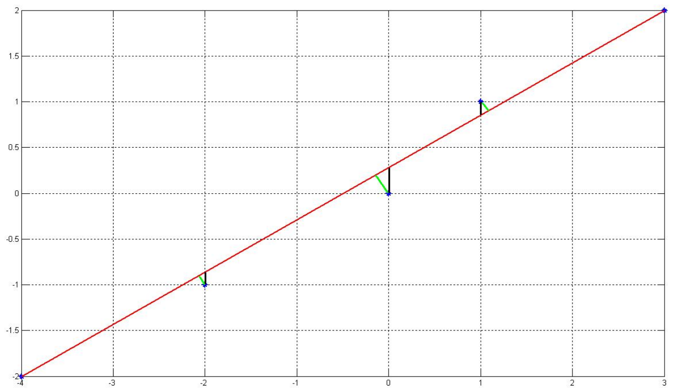

Figure 3: Minimizing the sum of the *perpendicular* distances is *not* the same as a MSE.

1. Find the mean of the point-set, the line should go through this point.
2. Calculate the covariance matrix. The line should be aligned with the eigenvector with the largest eigenvalue of the covariance matrix.

$$\bar{m} = \frac{1}{5}\sum_{i=1}^{5} p_i = \frac{1}{5}(-4-2+1+3, -2-1+1+2) = (-0.4, 0)$$

The covariance matrix Σ is given by:

$$\Sigma = \frac{1}{N-1}\sum_{i=1}^{N}\begin{pmatrix} (x_i - m_x)^2 & (x_i - m_x)(y_i - m_y) \\ (y_i - m_y)(x_i - m_x) & (y_i - m_y)^2 \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} 12.96 + 2.56 + 0.16 + 1.96 + 11.56 & 7.2 + 1.6 + 0 + 1.4 + 6.8 \\ 7.2 + 1.6 + 0 + 1.4 + 6.8 & 4 + 1 + 0 + 1 + 4 \end{pmatrix} =$$

$$\Sigma = \begin{pmatrix} 7.3 & 4.25 \\ 4.25 & 2.5 \end{pmatrix}$$

Now calculate the eigenvalues λ and the corresponding eigenvectors $\bar{v}$;

$$\det(\Sigma - \lambda I) = 0 \Rightarrow \begin{vmatrix} 7.3 - \lambda & 4.25 \\ 4.25 & 2.5 - \lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 9.8\lambda + 7.3 \cdot 2.5 - 4.25^2 = 0$$

$\Rightarrow \lambda_1 = 9.7808 \wedge \lambda_2 = 0.0192$

We can here clearly identify the largest eigenvalue, $\lambda_1 = 9.7808$. The eigenvector corresponding to this eigenvalue is obtained by solving the system of equations:

$$(\Sigma - \lambda_1 I)\,\bar{v} = 0 \Leftrightarrow \begin{pmatrix} 7.3 - \lambda_1 & 4.25 \\ 4.25 & 2.5 - \lambda_1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\frac{v_y}{v_x} = \frac{7.3 - 9.7808 - 4.35}{2.5 - 9.7808 - 4.35} = 0.5873$$

Since we are only interested in the direction of this eigenvector, the quotient $\frac{v_y}{v_x} = \frac{\Delta y}{\Delta x} = k$ is the only thing we're interested in. Now we know that our line goes through the point $\bar{m} = (-0.4, 0)$ and has a direction $k = 0.5873$. Thus, the equation of the line that minimizes the perpendicular distance sum to the point set is given by:

$$y - m_y = k(x - m_x) \Leftrightarrow y = 0.5873x + 0.2349$$

Excercise 3 (a), Exam 11/03/2003

An image has been smoothed with the following kernel:

$$h = k \cdot [1, 5, 10, 10, 5, 1]$$

Can repeated convolutions of an image with the kernel

$$g = \frac{1}{2}[1, 1]$$

be used to obtain the same result as with the first kernel? If yes, how many convolutions are needed? If no, explain the reasons why.

What should the constant k be so that the filter gain is equal to 1?

Answer: We see that $g * g = \frac{1}{4}[1, 2, 1]$, thus $g * g * g * g = \frac{1}{16}[1, 4, 6, 4, 1]$ and $g_*^5 = \frac{1}{32}[1, 5, 10, 10, 5, 1]$. Therefore, if $k = \frac{1}{32}$ we have $h = g_*^5$, i.e. five convolutions with the g kernel yields the h kernel for the given k-value.

Preparation for lab 3

(a) Derive a mask that approximates the first partial derivative in the x-direction when convolved with an image. **Answer:** $d_x = \frac{1}{2}[1, -1]$.

(b) Derive a mask, d_{xxx} for generating the third order derivative using the masks $d_x = 1/2(1, 0, -1)$, and $d_{xx} = (1, -2, 1)$ corresponding to the first and second order derivatives.

Answer: $d_{xxx} = d_x * d_{xx} = \frac{1}{2}[1, -2, 0, 2, -1]$.