Chapter 3

Mobile robot kinematics

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Mobile robot kinematics

Overview

- Wheel kinematic constraints
- Robot kinematic constraints
- Mobile robot maneuverability
- Mobile Robot Workspace
- Holonomic robots
- Path and trajectory considerations
- Beyond Basic Kinematics
- Kinematic Control

Kinematics

Introduction

- The mechanical behavior → Control
- Mobile robotics similar to Manipulator
- Unbound movement:
  - No direct way to measure
  - Position integration over time
  - Inaccuracy in position (because of mechanics)
- Each wheel:
  - Enabling
  - Constraints

Kinematics

Model:

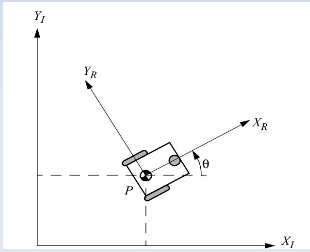
- Robot speed as a function of wheel speed
- Whole Robot's motion: a bottom-up process
- Chasis → Rigid body

Wheel Constraints

Robot Pos

- Global Reference
- Point P : Position Reference
- Local (Initial) Reference
- Position specified by:

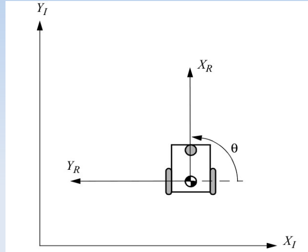
$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$



Forward Kinematics

- How does the robot move given its geometry and the speed of wheels?
- Orthoghonal conversion:
  - Map function

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta)\cdot\begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$$

$$R(\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

Example:

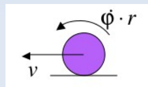
- Robot Pos: $\theta = \pi/2$, $r = 1$, and $l = 1$
- If: $\dot{\varphi}_1 = 4$ and $\dot{\varphi}_2 = 2$

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

Forward Kinematics

Wheel Kinematics

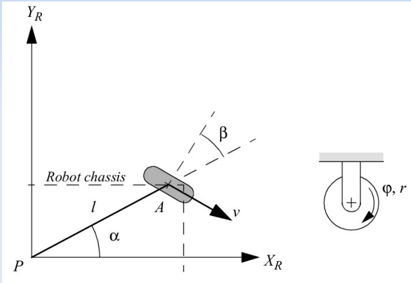
- Four types of wheels contraints
- Some simplifications:
  - Vertical plane for the wheel
  - Single point of contact (with no friction for rotation)
  - No sliding or sliding
  - Not deformable



Wheel Kinematics

Standard wheel

- No vertical axis of rotation → No steering
- A in polar coordinate
- The angle of wheel plane relative to chasis - B fixed

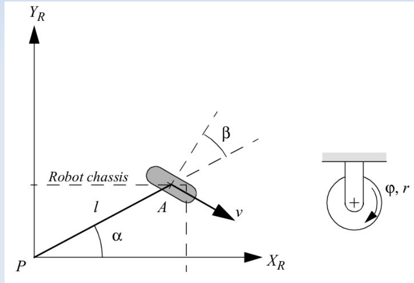


Wheel Constraints

Standard Wheel

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0$$



Standard Wheel

Example

- Suppose that the wheel A is in position such that α = 0 and β = 0
- This would place the contact point of the wheel on XI with the plane of
- The wheel oriented parallel to YI. If θ = 0, then this sliding constraint reduces to:

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = 0$$

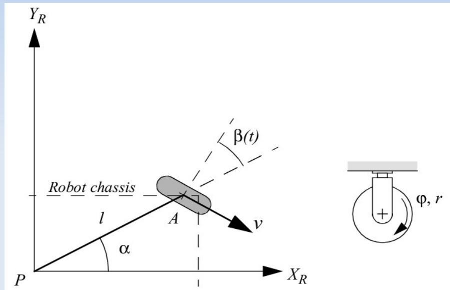
Wheels Constraints

Steered standard wheel

- Standard+rotation
- No instantaneous effect

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

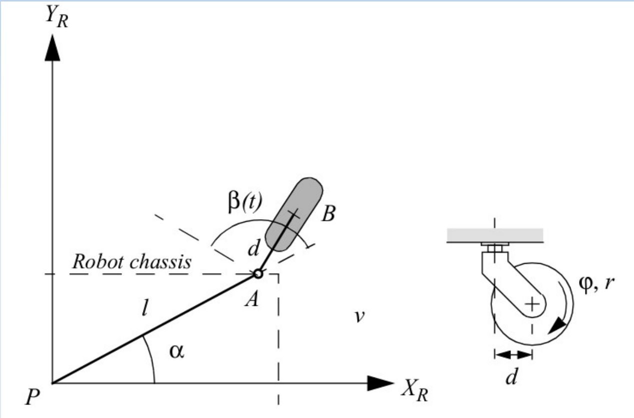
$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0$$



Wheels Constraints

Castor Wheel

- Steer around a vertical axis



Wheels Constraints

Castor Wheel

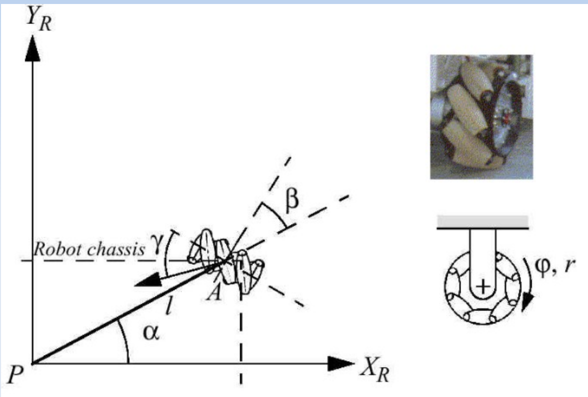
- Steer around a vertical axis
- Different vertical axis of rotation from contact point.
- Any motion orthogonal to the wheel plane must be balanced by and equivalent and opposite amount of castor steering motion

$$\left[\sin(\alpha+\beta)\ \ -\cos(\alpha+\beta)\ \ (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta)\ \ \sin(\alpha+\beta)\ \ d + l\sin\beta\right] R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$

Wheel Constraints

- Swedish wheel
  - Standard+1 DOF



Wheel Constraints

- Swedish wheel

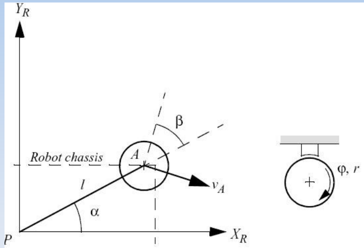
$$\left[\sin(\alpha+\beta+\gamma)\ \ -\cos(\alpha+\beta+\gamma)\ \ (-l)\cos(\beta+\gamma)\right] R(\theta)\dot{\xi}_I - r\dot{\varphi}\cos\gamma = 0$$

$$\left[\cos(\alpha+\beta+\gamma)\ \ \sin(\alpha+\beta+\gamma)\ \ l\sin(\beta+\gamma)\right] R(\theta)\dot{\xi}_I - r\dot{\varphi}\sin\gamma - r_{sw}\dot{\varphi}_{sw} = 0$$

Wheel Kinematic Constraints

Spherical Wheel:

- No direct constraints on motion.
- Has no principal axis of rotation so no appropriate rolling or sliding constraint exist.
- Omnidirectional
- No effects on robot chasis kinematics.
- The Eq. is similar to the fixed standard wheel but here the direction of movement is arbitrary



$$\left[\sin(\alpha+\beta)\ \ -\cos(\alpha+\beta)\ \ (-l)\cos\beta\right]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta)\ \ \sin(\alpha+\beta)\ \ l\sin\beta\right]R(\theta)\dot{\xi}_I = 0$$

Robot Kinematic Constraints

- Compute the kinematic constraints of a robot with M wheels.
- Combine the constraints that arise from all the wheels based on the placement of them on the robot chassis.
- Only standard fixed and steering wheels have constraints.

Robot Kinematic Constraints

- N wheels N_f + $N_{s.}$
- **The Rolling constraint:**

  It is the constraint that all standard wheels must spin around their horizontal axis an appropriate amount based on their motions along the wheel plane so that rolling occurs at the ground contact point.

$$J_1(\beta_s)R(\theta)\dot{\xi}_I - J_2\dot{\varphi} = 0$$

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

$$J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}$$

- **The Sliding constraint:**

  The components of motion orthogonal to the wheel planes must be zero for all standard wheels.

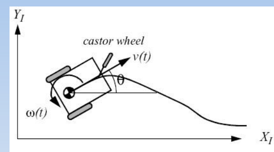
$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

Sliding constraint in standard wheels has the most significant impact on defining the overall maneuverability of the robot chassis.

Robot Kinematic Constraints

- The combination of wheel rolling and sliding constraints describes the kinematic behaviour.
- ***Example:*** A differential-drive robot.
- By defining alpha and beta angles for both wheels, J_{1f} and C_{1f} matrices can be computed.



$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix}$$

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2l} & -\frac{1}{2l} & 0 \end{bmatrix} \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix}$$

Mobile Robot Maneuverabiity

- Kinematic mobility: Robots ability to directly move in the environment.
- The basic constraint in mobility is satisfying the sliding constraint.

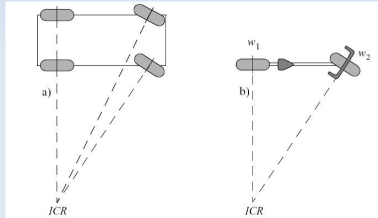
Mobile Robot Maneuverabiity

- **Degree of Mobility:**
  - For both of these constraints to be satisfied, the motion vector *R* (θ)ξ_1*_dot* must belong to null space of the projection matrix C_1 (β_s)

$$C_{1f}R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}(\beta_s)R(\theta)\dot{\xi}_I = 0$$

- **Instantaneous center of Rotation:**
  - Those equations can be represented geometrically by ICR.
  - Zero motion line. Perp to wheel plane
  - ICR geometric construction demonstrates how robot mob. is a function of the # of the constraints not the # of wheels.
  - Robot chasis kinematics is therefore a function of the set of **indipendent constraints**. Arising from all standard wheels.



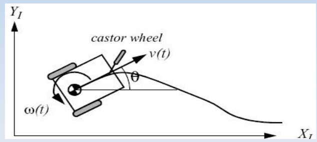
Mobile Robot Maneuverabiity

- The *rank[C_1(β_s)] is the # of indipendent constraints. (Give the exp. of rank of a matrix.)*
- *More rank = more constraints in mobility.*
- *Robot with single fixed standard wheel is rank 1*
- In general robot will have 0≤ *rank* C_1(β_s)≤ 3
- Extreme cases?

Mobile Robot Maneuverabiity

- **Degree of mobility(δ_m)**
  - It is a measure of the # of DoF of robot chassis that can be immediately manipulated through changes in wheel vel.
  - Example of differential drive robot: DoM=2
  - Example of bicycle. DoM=1

$$\delta_m = dimN\left[C_1(\beta_s)\right] = 3 - rank\left[C_1(\beta_s)\right]$$



- **Degree of steerability(δ_s)**
  - Increase in DoS results eventually greater maneuverability but decrease mobility.
  - Range 0≤ δ_s≤ 2

$$\delta_s = rank\left[C_{1s}(\beta_s)\right]$$

- **Robot maneuverability.**
  - $\delta_M = \delta_s + \delta_m$.

Mobile robot workspace

- How can a robot use its control degrees of freedom to position itself in the environment?
- What are the possible trajectories that a robot can follow?

The answer is related to the robots *Degrees of Freedom* (DoF) and *Differentiable Degrees of Freedom* (DDoF)

Mobile robot workspace

- Differentiable Degrees of Freedom (DDoF) affect the ability of the robot to achieve various paths
- Degrees of Freedom (DoF) affect the ability of the robot to achieve various poses

Mobile robot workspace

- Differentiable Degrees of Freedom (DDoF)

DDoF = δ_m (degree of mobility)

Example:

Bicycle -> $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$

DDoF = 1 but DoF = 3

Omnibot -> $\delta_M = \delta_m + \delta_s = 3 + 0 = 2$

DDoF = 3 and DoF = 3

$$DDOF \leq \delta_M \leq DOF$$

Holonomic robots

- In mobile robotics, the term refers specifically to the *kinematic constraints* of the robot chassis
- A *holonomic* robot has *zero* non-holonomic kinematic constraints

Holonomic robots

- A *holonomic kinematic constraint* can be expressed as an explicit function of position variables only.
- A *non-holonomic kinematic constraint* requires a differential relationship and it cannot be integrated to provide a constraint in terms of the position variables only

Holonomic robots

Example:

Let's consider a bicycle with a locked front wheel

$$\delta_M = 1$$

and

$$[-\sin(\alpha+\beta)\cos(\alpha+\beta)\ l\ \cos\beta\ ]\ R(\theta)\xi l + r\dot{\phi} = 0$$

which can be replaced by

$$\phi = (x/r) + \phi_0$$

therefore this bicycle is holonomic!

Holonomic robots

A more intuitive way to describe holonomic robots is to say that:

DDoF = DoF

must hold.

In general we require DDoF = DoF = 3,

meaning that we 'prefere' omnidirectional robots

Path and trajectory

Although we like holonomic robots, there are some serious considerations:

- Their design is more complex and expensive
- They are less stable during movement

Consider the *Omnibot*!

Beyond Basic Kinematics

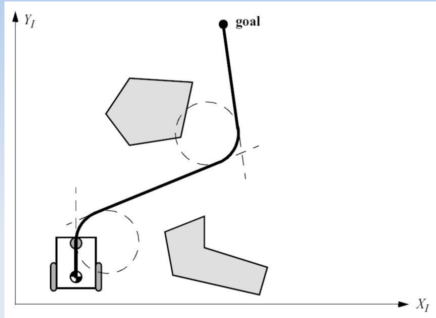
More things are to be considered in real life:

- Dynamic constraints due to speed and forces
- Violation of the previously defined kinematic models
- Presence of friction
- Actuation of the available degrees of freedom

Need for control systems!

Kinematic Control

- Open loop control
  (trajectory following)
  - Not always easy to find a feasible trajectory that meets the constraints
  - Not smooth trajectories
  - Not adaptive to changing environments



Kinematic Control

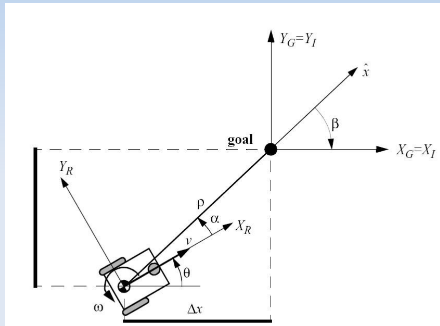
- Feedback control
  - Define the kinematic model of the robot
  - Find a control matrix K such that the robot moves to the desired position
  - Use of K must result in a 'stable' system

Kinematic Control

- <u>Example</u>

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \, {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \quad \text{with } k_{ij} = k(t, e)$$



Kinematic Control

- Example

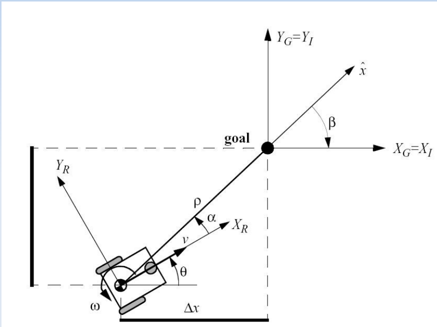
Mobile robot model

$$ {}^{I}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} $$

$$ \rho = \sqrt{\Delta x^2 + \Delta y^2} $$

$$ \alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x) $$

$$ \beta = -\theta - \alpha $$



Kinematic Control

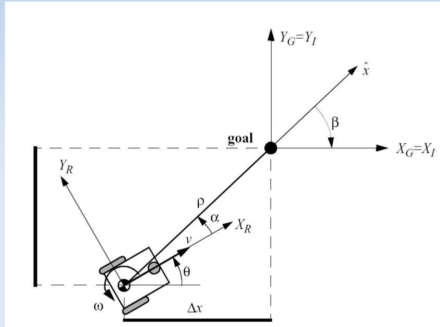
- Example

$$ \begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}\cos\alpha & 0\\ -\dfrac{\sin\alpha}{\rho} & 1\\ \dfrac{\sin\alpha}{\rho} & 0\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} $$

Control law

$$ v = k_\rho \rho $$

$$ \omega = k_\alpha \alpha + k_\beta \beta $$



Kinematic Control

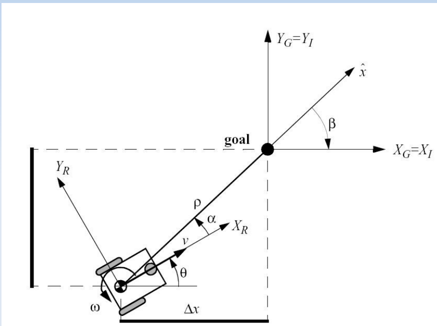
- Example

Closed loop system

$$ \begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-k_\rho \rho \cos\alpha\\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta\\ -k_\rho \sin\alpha\end{bmatrix} $$

Unique equilibrium point at

$(\rho,\alpha,\beta) = (0, 0,0)$



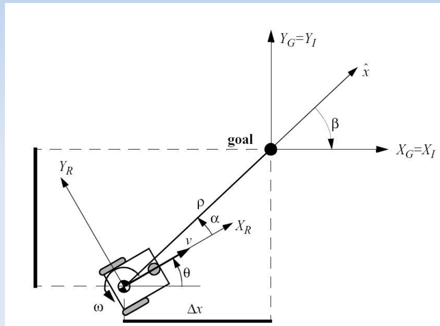
Kinematic Control

- Example

Stable if:

$k_\rho > 0, \quad k_\beta < 0, \quad k_\alpha - k_\rho > 0$

$$ A = \begin{bmatrix}-k_\rho & 0 & 0\\ 0 & -(k_\alpha - k_\rho) & -k_\beta\\ 0 & -k_\rho & 0\end{bmatrix} $$



Mobile robot kinematiks

Thank you!

Questions ???