LAST TIME

Exponential lower bound on resolution refutations on pigeonhole principle formulas

\[ L_R(\text{PHP}^{n+1}_n - 1) = \exp(\Omega(n)) \]

\( \text{PHP}^n_n \) has size \( N = \Theta(n^3) \)
So lower bound \( \exp(\Omega(\sqrt[3]{N})) \)
in terms of formula size.

Proof idea (high level)

Look at Prosecutor-Defendant game
Good strategies for Defendant: resolution lower bounds
Defendant picks random matching of \( n/4 \) pigeons to \( n/4 \) pigeonholes
Exponentially many different choices.
Before Prosecutor wins, has to work down noticeable fraction of
information about this random matching
\( \Rightarrow \) exponentially many records
\( \Rightarrow \) exponential resolution lower bound

TODAY

Look at formulas encoding (contradiction of)
handshaking lemma: sum of vertex degrees in
undirected graph = even number = 0 (mod 2)
Obtain truly exponential lower bounds \( \exp(\Omega(N)) \).
Prosecutor-Defendant Game [Pudlák '00]

Unsatisfiable CNF formula $F$

Prosecutor maintains record $R = \text{partial truth value assignment to } \text{Vars}(F)$

Every record $R$ has an instruction of type:

(a) ask about $x$

Defendant answers $6 \in \{0, 1\}$; prosecutor adds $x = 6$ to $R$

(b) Forget values, i.e., shrink assignment to $R' \not\subseteq R$

A winning position $R$ for prosecutor is an assignment falsifying some $C \in F$

A (complete) strategy $S$ for prosecutor is a list of record such that

- there is a record for every possible defendant response
- regardless of how defendant plays, prosecutor always wins in the end.

The size of a strategy $S$ is the number of records in it.

**Lemma** [Pudlák '00]

$F$ has a resolution refutation in length $L$ iff prosecutor has a strategy of size $\Theta(L)$

Proof ($\Rightarrow$) We did this last lecture

($\Leftarrow$) Exercise (not superhard, but not trivial either)
Recall notation $x^y$ for literals

$x^1 = x \quad x^0 = \overline{x}$

$x^6$ literal satisfied by setting $x = 0$

\[ |V| = |V(G)| = n \]

**Tseitin Formula**

Let $G = (V, E)$ a undirected graph

$\chi : V \rightarrow \{0, 1\}$ charge function

Identity every edge $e \in E$ with variable $x_e$

Let $\text{PARITY}_{\chi, x}$ be set of clauses encoding that the number of true edges incident to $v$ is equal to $\chi(v) \pmod{2}$ — i.e., the parity of the sum is $\chi(v)$

$$\text{PARITY}_{\chi, x} = \sum_{e \in v} x_e = \chi(v) \pmod{2}$$

In more formal notation, this is the set of clauses

$$\text{PARITY}_{\chi, x} = \left\{ \bigvee_{e \in v} x_e \mid \sum_{e \in v} x_e \neq \chi(v) \pmod{2} \right\}$$

For every assignment to $\{x_e \mid e \in V\}$ that gets the parity **wrong**, i.e., $\sum_{e \in v} x_e \neq \chi(v) \pmod{2}$ at least one literal $x^{1-_b} e$ has to be satisfied in the clause, showing that we do not have this assignment

The Tseitin formula over $G$ with respect to $\chi$ is

$$T_{\chi}(G, \chi) = \bigwedge_{v \in V} \text{PARITY}_{\chi, x}$$
If $G$ has bounded degree $d$ (i.e., all vertices have at most $d$ incident edges), then $TS(G, \chi)$ is a $d$-CNF formula with at most $2^{d-1} |V|$ clauses and $dN^{1/2}$ variables.

When talking about Tseitin formulas in what follows, we always assume that $G$ has bounded degree $d = O(1)$ unless stated otherwise.

Say that $\chi: V \to \{0,1\}$ is an odd-charge function if $\sum_{v \in V} \chi(v) = 1 \pmod{2}$.

**Proposition 1**

If $G$ is connected, then $TS(G, \chi)$ is unsatisfiable if and only if $\chi$ is an odd-charge function.

Proof. Exercise.

In what follows, always assume graph $G$ underlying Tseitin formula is connected unless stated otherwise.
Today we want to prove that if $G$ is a well-connected graph, then resolution refutations of $T_c^*(G, \phi)$ require exponential length (truly exponential $\exp(-\Omega(N))$ in size $N$ of formula).

Two comments:
- For even-charge $\chi$ this is obviously true (satisfiable formulas are very hard to refute).
- For odd-charge $\chi$ we only really care about the charge. All odd-charge functions on $V$ are equivalent from the point of view of resolution. (Won't need this; it is just a side note.)

Well-connected graphs are known as expanders. Several different ways of measuring expansion:

1. **Vertex expansion**
   Every small-to-medium vertex set $U \subseteq V(G)$ has many neighbors in $N(U) \setminus U$.

2. **Edge expansion**
   Every small-to-medium-large $U \subseteq V$ has many outgoing edges to $V \setminus U$.

3. **Algebraic expansion**
   The gap between the two largest eigenvalues in the (normalized) adjacency matrix is large.
These notions are all tightly connected (but proving this is out of scope for this course). We will use edge expansion.

For $G = (V, E)$ and $U \subseteq V$, let $\partial U$ denote the set of outgoing edges from $U$, i.e.,

$$\partial U = \{(u, v) \in E \mid u \in U, v \in V \setminus U\}.$$

**Definition 2 (Edge expansion)** An undirected graph $G = (V, E)$ is a $(d, \delta)$-edge expander if for every vertex set $U \subseteq V$ of size $|U| = N/2$ it holds that $|\partial U| \geq \delta |U|$.

That is, a constant fraction of edges incident to $U$ are exiting $U$.

Now we can state the goal of today’s lecture.

**Theorem 3** [Urughart ’87]

Fix $d \in \mathbb{N}^+$, $\delta > 0$, and suppose that $\{G_n\}_{n=3}^{\infty}$ is a family of $n$-vertex, $(d, \delta)$-edge expanders. Then for any family of odd-charge functions $X_n : V(G_n) \to \{0, 1\}$ it holds that the \textsc{CNF} formula family $\{Ts(G_n, X_n)\}_{n=3}^{\infty}$ requires resolution refutations of length $\exp(-2(n))$.

**Remarks:** (i) Formula size is $O(n)$, so truly exponential

(ii) Concrete constants will depend on $d, \delta$. 
Will not do Urgaun's proof, but our own Prosecutor-Defendant-style proof (cooked up together with Massimo Lauria and Per Austrin, but any errors are the responsibility of the lecturer, of course) slightly weaker result: require expansion $\delta \geq 1$.

But first a natural question:
Is this a non-vacuous theorem? i.e., are there such edge expanders?

Yes. Picking a random $d$-regular graph will do.

**Theorem 4** [Bollobás 88]
Fix $d \in \mathbb{N}^+$, $d \geq 3$. Then there exists a universal constant $\delta > 0$ such that asymptotically almost surely a random $d$-regular graph is a $(d, \delta)$-edge expander.

A family of events $\{E_n\}_{n=1}^\infty$ happens asymptotically almost surely (a.a.s.) if

$$\lim_{n \to \infty} \Pr[ E_n ] = 1$$

Sometimes also referred to as "with high probability" (whp) but we will try to stick to a.e.s. in this course.

In fact, also possible to give explicit constructions of families of expander graphs, but these results are highly nontrivial.
MORE ABOUT EDGE EXPANSION

The maximal edge expansion of a graph is known as the ISOPERIMETRIC NUMBER \( h(G) \) (or the CHEEGER CONSTANT), i.e.,

\[
h(G) = \min_{U \subseteq V, |U| \leq |V|/2} \frac{|\partial U|}{|U|}
\]

For a random graph and any subset \( U \subseteq V(G) \) of size \( |U| \leq |V(G)|/2 \), we would expect roughly half of the edges in \( U \) to go to \( V \setminus U \). So a random \( d \)-regular graph might have edge expansion something like \( d/2 \) if we are lucky. This is indeed the case for \( d \) large enough.

**THEOREM** [Bollobás '88]

For every \( \varepsilon > 0 \) there is a \( d \in \mathbb{N}^+ \) such that a random \( d \)-regular \( n \)-vertex graph has edge expansion at least \( \frac{d}{2} - \varepsilon \) asymptotically almost surely as \( n \to \infty \).

Bollobás actually calculates a more precise expression from which it follows that random 6-regular graphs have edge expansion at least \( \frac{1}{2} \) a.a.s. and random 4-regular graphs have expansion 0.4 a.a.s.5.
In the proof, will also be convenient to use another (fourth) notion of expansion.

**Definition 5** (Connectivity expansion)
An undirected graph \( G = (V, E) \) is a \((d, \alpha)\)-connectivity expander if it has bounded degree \( d \) and for every edge set \( E' \subseteq E \), \(|E'| \leq \alpha n\), it holds that the graph \( G' = (V, E \setminus E') \) has a connected component of size strictly greater than \(|V|/2\).

**Proposition 6**
Every \((d, \delta)\)-edge expander is a \((d, \alpha)\)-connectivity expander for \( \alpha = \delta/4 \).

Proof. Exercise.

To save typing in what follows, let us say that any edge set \( E' \) of size \(|E'| \leq cn\) is of "moderate size" and that the (unique) connected component of size \(>|V|/2\) is "the large component".

For any moderate-size \( E' \) an assignment \( g : E' \to \{0, 1\} \) is "charge-preserving" if in \( G' = (V, E \setminus E') \) the large component has odd charge and any small component has even charge.

(Charge function \( \chi \) updated for \( G' \) by plugging in values from \( g \))
**Example**

\[ S_2 = \{ y \rightarrow 1 \} \quad S_2 = \{ x \rightarrow 1, y \rightarrow 1, w \rightarrow 0 \} \]

**Observation**

Where the odd charge ends up only depends on the parity of the cut.

**Defendant Strategy** (a bit high level)

\( G = (V,E) \) is a \((d, \delta)\)-edge expander and \((d, c)\)-connectivity expander for \( c = \delta/4 \).

Pick uniformly at random a set \( E' \subseteq E \) of size \( cn/k \).

Sample uniformly at random a charge-preserving assignment to \( E' \).

**NB!** Different edges are not set independently. E.g., above, if \( x, u \in E' \), then once we have randomly set \( x \) we must set \( u = x \).

But many edges in \( E' \) will be set independently at random (need to argue this carefully).

Call this assignment \( \text{Def} \).

Defendant will maintain \( g \) such that:

1. \( g \geq g_{\text{init}} \)
2. \( g \) consistent with \( \mathcal{R} \)

In fact, will have \( g = g_{\text{init}} \cup \mathcal{R} \).
When Prosecutor asks about value of $x_e$ (=edge):

(i) If $e \in \text{Dom}(g)$, answer according to $g(x_e)$
   [In this case $e \in \text{Dom}(g_{\text{fin}})$, otherwise
   Prosecutor knows the answer and wouldn't ask]

(ii) If $e \notin \text{Dom}(g)$, answer with value $0 < \epsilon < 1/3$
    so that the odd change in
    $G' = (V, E \setminus (\text{Dom}(g) \cup \{e\}))$
    stays in the
    large component.

If this is not possible (since the large component
    disappeared) give up, or answer arbitrarily
    or whatever.

When Prosecutor forgets to get $R' \notin R$:

Simply update $g$ to $g = g_{\text{fin}} \cup R' - R$

Now we want to implement the same
lower bound strategy as for PHP

(1) Before Prosecutor wins, has to have
    "informative record" containing lots of edges

(2) Such records contain lots of information
    about Defendant's initial random choice

(3) Hence, any given informative record is
    exponentially unlikely to be consistent with
    a particular random choice

(4) So strategy contains exponentially many records.
Observation 7

Before Prosecutor wins there must be a record with more than $cn/2$ edges.

Proof. A winning position for Prosecutor is a falsified vertex constraint = an odd-
charge disconnected component of size 1.

Let $E'$ initial random choice by Defendant.

Let $E''$ be edges in Prosecutor record.

As long as $|E' \cup E''| \leq cn$, Defendant is making sure the odd charge is in the large
component. Hence, before Prosecutor wins we must have $|E' \cup E''| = |E''| + |E' \setminus E''|$

$\geq cn$ or $|E''| \geq cn - |E' \setminus E''| \geq cn/2$. $\square$

Call a record with $\geq cn/2$ edges informative.

We want to prove that a fixed informative record $R$ has exponentially small probability of
being consistent with Defendant's initial random choice $E'$.

For any edge $e \in R$, $\Pr[e \in E'] = \frac{cn/2}{\# edges |E(G)|} \geq \frac{cn/2}{dn/2} = \frac{c}{d}$

By linearity of expectation

$\mathbb{E} \left[ \sum_{e \in \text{Dom}(R) \cap E'} \right] \geq \frac{cn}{2} \cdot \frac{c}{d} = \frac{c^2}{2d} \cdot n$
By concentration of measure (same calculating as for PHP) it holds that
\[
\Pr \left[ |\text{Dom}(R) \cap \mathcal{E}'| \leq \frac{c^2}{4d} n \right] = 2^{-\Omega(n)} \tag{1}
\]
for some \( \epsilon > 0 \) (for \( n \) large enough).

Fix \( \mathcal{E}_1 = \text{Dom}(R) \cap \mathcal{E}' \) and assume for now that
\[
|\mathcal{E}_1| \geq \frac{c^2}{4d} n
\]
By construction also have
\[
|\mathcal{E}_1| \leq |\mathcal{E}'| = cn/2
\]

**LEMMA 8** (Key Technical Lemma)

Suppose that \( G \) is a \((d, \delta')\)-edge expander for \( d > 1 \) and let \( \mathcal{E}_1 \subseteq \mathcal{E}(G) \) be any moderate-size edge set (i.e., \( |\mathcal{E}_1| \leq cn^2 \) for \( c = 5/4 = 1.25 \)).

Then there is a subset \( \mathcal{E}_2 \subseteq \mathcal{E}_1 \) of size
\[
\geq y |\mathcal{E}_1|
\]
for some \( y > 0 \) such that if \( \mathcal{E}_2 \) is a uniformly randomly sampled charge-packing assignment to \( \mathcal{E}_1 \), it holds that all edges in \( \mathcal{E}_2 \) are assigned uniformly and independently at random.
Taking Lemma 8 on faith for now, we can prove Theorem 3 (for $\delta \geq 1$).

**Proof of Theorem 3**

Let $S$ be a complete strategy for prosecutor for $TS(G_n, \mathcal{X}_n)$ for $n$ large enough.

Any game goes through informative record $R$ with probability $1$. This $R$ is consistent with $g_{\text{mir}}$.

If we can prove for any fixed informative $R$ that
\[
\Pr \left[ R \& g_{\text{mir}} \text{ consistent} \right] \leq 2^{-\varepsilon n}
\]
it follows that size of $S > \# \text{ informative records in } S \geq 2^\varepsilon n$

\[
\Pr \left[ R \& g_{\text{mir}} \text{ consistent} \right] \leq \varepsilon
\]

\[
\Pr \left[ |\text{Dom}(R) \cap \mathcal{E}'| \leq \frac{e^2}{4d} n \right] + \varepsilon
\]

\[
\Pr \left[ R \& g_{\text{mir}} \text{ consistent} \right] \left[ |\text{Dom}(R) \cap \mathcal{E}'| \geq \frac{e^2}{4d} n \right]
\]

By (†) above we have (‡) $\leq 2^{-\varepsilon n}$

For (‡) we have that $\gamma |\mathcal{E}'| \geq \gamma \frac{e^2}{4d} n$ edges are set uniformly and independently at random.

Agreement with $R$ with probability
\[
\left( \frac{1}{2} \right)^{\frac{e^2}{4d} n} = 2^{-\varepsilon n}
\]
Combining this we get
$\Pr \left[ E \text{ is } \theta \text{-min cut} \right] \leq 2^{-\varepsilon n} + 2^{-\varepsilon'' n} \leq 2^{-\varepsilon n}$
for some $\varepsilon > 0$ and Theorem 3 follows.

Lemma 8 follows from the following two lemmas:

**Lemma 9**

For $G = (V, E)$, a $(d, \delta)$-expander with $\delta > 0$, suppose that $E_1 \subseteq E(G)$ is a moderate-size set and that $E_2 \subseteq E_1$ does not disconnect $G$ (i.e., $(V, E \setminus E_2)$ is a connected graph). Then uniformly sampling of a charge-preserving assignment to $E_1$ gives a uniformly random sample of $\{0, 1\}^{E_2}$, for the edges in $E_2$.

(We will use the short-hand $\{0, 1\}^{E_2}$ for the set of all possible assignments to edges in $E_2$.)

**Lemma 10**

Let $G = (V, E)$ be a $(d, \delta)$-expander with $\delta \geq 1$ and let $E_1 \subseteq E$ be any moderate-size set. Then there is a subset of edges $E_2 \subseteq E_1$, of size $|E_2| = \Omega(|E_1|)$ such that $E_2$ does not disconnect $G$.

i.e., there is a global constant $\gamma > 0$ such that

$|E_2| \geq \gamma |E_1|$
Proof of Lemma 8

Consider the set $E_1$ in Lemma 8, which is of moderate size. Lemma 10 guarantees existence of $E_2$ of size $|E_2| > y |E_1|$ for some $y > 0$ such that $G' = (V, E \setminus E_2)$ connected.

Now Lemma 9 says when we randomly sample a charge-preserving assignment to $E$, we get uniform and independent random 6/5 in $E_2$.

Lemma 9 is mostly some linear algebra juggling and will probably appear on post 1. Let us do Lemma 10 first.

Proof of Lemma 10

Let $E_1 \in E(G)$ be any moderate-size set, i.e. $|E_1| \leq cn$ for $c = 5/4$.

Look at the connected components in $G' = (V, E \setminus E_1)$. Let sum of their sizes is $S$.

Case 1. $S \leq |E_1|/2d$

(Sum of sizes of small components is at least small.)

Then the total # edges not completely inside the large component is at most $Sd \leq |E_1|/2$.

So if we pick $E_2$ to be the edges in $E$, between vertices in the large component we have $|E_2| \geq |E_1|/2$. 

XU
Case 2 \[ S > \frac{|E,1|}{2d} \]

The sum of the sizes of the small components is larger, but not too large (by connectivity expansion)

Remove from \( E \) any edges not incident to the large component (black edges) to get \( E^* \leq E \). This might merge some small components, but they are still smallish and the total size \( S \) stays the same.

All edges in \( E^* \) go from small components to the large component. By edge expansion,

\[ |E^*| > \delta S > \delta \frac{|E,1|}{2d} \]

Fix one edge per (merged) small component going to the large component (red edges). Let \( E^{**} = E^* \setminus \{ \text{these red edges} \} \). Since \( \delta > 1 \) every small component has \( \geq 2 \) edges to the large components, so red edges are at most half of \( E^* \). Remaining edges in \( E^{**} \) are the green edges.
By construction, \( G_{**} = (V, E \setminus E^{**}) \) is connected and
\[
|E^{**}| \geq |E^*|/2 > \frac{3|E|}{4d} = \Omega(1E,1)
\]
The lemma follows.

It remains to prove Lemma 9 that if \( E' \) does not disconnect \( G \), then change-preserving assignments to a superset \( E'' \supset E' \) yields uniform random bits on \( E' \). We will mostly leave this as a linear algebra exercise for the first problem set, but provide some useful background facts in an appendix to these notes.
Suppose that $V$ is a vector space over some field $F$. From before, you might be used to $F$ being the real numbers $\mathbb{R}$ or the complex number $\mathbb{C}$, but here we will have $F = GF(2)$, i.e., the field with two elements 0, 1 such that:

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 1 = 0$
- $0 \cdot f = 0$
- $1 \cdot f = f$ for $f \in \{0, 1\}$

Most of the basic facts about vector spaces still hold in this setting.

An affine subspace $A$ of $V$ is a set $A = \{ a + u | u \in U \}$ for some fixed $a \in V$ and some fixed (linear) subspace $U \leq V$.

That is, an affine space is a linear space shifted by a constant vector.

If $u, v \in A$, then $u - v \in A$, but in general $u + v$ might not be in $A$.

If always $u, v \in A$ implies $u + v \in A$ then $A$ is linear, i.e., we can choose the offset $a = 0$.

For simplicity, in what follows let us focus on affine subspaces of $\{0, 1\}^m = GF(2)^m$. 
FACT A

Any affine subspace $A \subseteq \{0, 1\}^m$ of dimension $n \leq m$ is generated by $Mx + \alpha$, for some fixed (but not unique) $n \times m$ matrix $M$ of (full) rank $n$ and some fixed size-$m$ column vector $\alpha$.

When we let $x$ range over (all column vectors in) $\{0, 1\}^n$. Uniform random sampling from $A$ can be performed by choosing a uniformly random $x \in \{0, 1\}^n$.

We will just accept this as true—you can take it as a definition if you like.

PROPOSITION B

Let $A \subseteq \{0, 1\}^m$ be an affine subspace. Suppose for a subset of coordinates $S \subseteq [m]$ that all bitstrings in $\{0, 1\}^S$ are supported by $A$ (i.e., there are vectors $u \in A$ that agree with any bit pattern in $\{0, 1\}^S$).

Then a uniformly random sample from $A$ yields uniformly random and independent bits when restricted to $\{0, 1\}^S$.

Proof Exercise, but let us hint at two possible solutions.
Approach 1
Argue that (by the way we have defined things) in any affine subspace \( A \subseteq \{0,1\}^m \) any bit that is not fixed to 0 or 1 is 0 in exactly half of the vectors and 1 in exactly half of the vectors (why?)

Repeat this on bit after bit in \( S \) using that fixing a bit yields another affine subspace \( A' \subseteq A \) (why?)

Argue that every fixed bit pattern in \( \{0,1\}^S \) must appear in a fraction \( 1/2^{||S||} \) of the vectors in \( A \).

Approach 2
The rows \( \{ R_i \} \ i \in S \) in \( M \) must be linearly independent. (Why?)

This means that the submatrix consisting of these rows has rank \( ||S|| \), and hence there exists a set of \( T \) columns, \( |T| = ||S|| \), such that the submatrix on rows \( S \) and columns \( T \) is invertible. Argue that for any choice of the values of \( x \) outside of the coordinates in \( T \) the values of \( Mx + a \) over \( \{0,1\}^T \), restricted to the coordinates in \( S \), is one-to-one and hence uniform over random \( x \).
Phrased differently, Proposition B says that if an affine subspace is supported on the uniform distribution of some set of coordinates $S$, then sampling from $A$ uniformly at random and restricting to the coordinates in $S$ yields the uniform distribution over $[0,1]^S$.

**Observation C**

Let $G = (V,E)$ be any connected graph, let $X : V \to \{0,1\}$ be any odd-chance function, and let $E' \subseteq E$ be a set of edges such that $G' = (V,E \setminus E')$ has a (unique) connected component of size $\geq |V|/2$. Then the set of charge-preserving assignments $A \subseteq \{0,1\}^{E'}$ form an affine subspace.

To prove this we need another observation.

**Observation D**

Let $G = (V,E)$ be a connected graph with an odd-chance function $X$ and let $E' \subseteq E$ be a minimal set of edges that disconnects $G$ into two connected subgraphs $G_1$ and $G_2$. Then charges of the subgraphs $G_1$ and $G_2$ resulting from any assignment $g : E' \to \{0,1\}$ depends only on $X$ and on the parity of $\sum_{e \in E'} g(e)$.

Proof Exercise.
Using Observation D it is straightforward, if a bit tedious, to prove Observation C.

Proof of Observation C (sketch)

Consider $G' = (V, E' \setminus E')$. Let $G_0$ be the unique large component and $G_i$, $i = 1, \ldots, s$, the small connected components. Let $E_{ij}$ be the edges between $G_i$ and $G_j$.

Look first at $G_0 = (V, E \setminus \bigcup_{i=1}^{s} V_i, E_{0,j})$, small. This yields one affine constraint per connected component in $G_0$ requiring that the odd charge is pushed into the large component.

Now consider $E_{ij} = 0$ for $1 \leq i < j \leq s$.

If adding $E_{ij}$ to the set of previously considered edges adds a new connected component, i.e., splits one small component into two smaller ones, the constraint that both new small components should get even charge is an affine constraint on the edges $E_{ij}$ and the previously assigned edges. All of these constraints can be collected in matrix form $M^* y = b$, and the set of solutions can be written as $M x + b$ for some $x$ of suitable dimension. 

\[\]
Proposition 8.11

Suppose \( G = (V, E) \) is a connected graph with an odd charge function \( \chi \) and let \( E_1 \subseteq E \) be such that \( G_1 = (V, E \setminus E_1) \) has a unique connected component of size \( > |V|/2 \). Then for any fixed \( E_2 \subseteq E \), the set of charge-preserving assignments to \( E_1 \) has full support on \( [0, 1]^{E_2} \) if and only if \( G_2 = (V, E \setminus E_2) \) is connected.

Proof (\( \Leftarrow \)) If \( G_2 \) is connected, then clearly we can assign all edges \( e \in E_2 \) arbitrarily, since there is only one component and its charge stays the same. Assignments to edges in \( E \setminus E_2 \) will take care of charges when the graph is disconnected. Hence we have full support on \( [0, 1]^{E_2} \).

(\( \Rightarrow \)) Pick a minimal set \( E^* \subseteq E_2 \) that disconnects \( G \). The parity of this set of edges must be such that the odd charge stays in the large component. Hence we can only have half of the assignments to \( [0, 1]^{E^*} \) and, in particular, do not have full support on \( [0, 1]^{E_2} \).
Now it is straightforward to prove Lemma 9 in the notes for Lecture 3, stated here again for reference.

**Lemma 9**

For $G = (V, E)$ a $(d, \delta)$-expander with $\delta > 0$, suppose that $E_1 \subseteq E$ is a moderate-size set and that $E_2 \subseteq E_1$ does not disconnect $G$. Then uniformly random sampling of charge-preserving assignments to $E_1$ yields uniformly random samples of $E_1 \subseteq E_2$.

**Proof** An exercise in putting together the facts, propositions and observations we have covered so far.