KTH Computer Science
and Communication

Homework III, Complexity Theory Fall 2011

Due on September 22 at 15.15, i.e. at the beginning of the lecture. The general rules on homework solutions available at the course home-page apply. In particular, discussions of ideas in groups of up to at most two people are allowed but solutions should be written down individually.

Some of the problems below are "classical" and hence their solutions is probably posted on the Internet. It is not allowed to use such solutions in any way. The order of the problems is "random" and hence do not expect that the lowest numbered problems are the easiest.

Any corrections or clarifications on this problem set will be posted under "homework" on the course home-page http://www.csc.kth.se/utbildning/kth/kurser/DD2446/kplx11/uppgifter.

Try to provide correct proofs to the problems below and refrain from imprecise statements.

Note that this problem set is unusually large. As the rules for the grades are not changed, this is in fact to the advantage of the student.

1 (10p) Make a resolution proof for pigeon-hole principle that three pigeons cannot fit into two holes. This formula has 6 variables, x_{ij} for $1 \leq i \leq 3$ and $1 \leq j \leq 2$ where the intuitive meaning is that x_{ij} is true iff pigeon number i sits in hole j. The clauses are that each pigeon should sit somewhere, i.e. $(x_{i1} \vee x_{i2})$ for $1 \leq i \leq 3$ and that no two pigeons sit in the same hole, i.e. $(\bar{x}_{i_1 j} \vee \bar{x}_{i_2 j})$ for $i_1 \neq i_2$ and $j = 1, 2$.

2 (15p) Prove that resolution is complete. That is, to prove that for any CNF-formula that is not satisfiable, there is a resolution proof that ends with the empty clause.

Hint: There are a couple of proofs of this fact but one simple approach can be to use the correspondence between resolutions proofs and the decision tree searching for a falsified clause. In the latter you are given a nonsatifiable CNF-formula and you can query variables for their values and must eventually come up with a falsified clause. As the formula is not satisfiable this must be possible for any assignment to the variables. Consider the entire decision tree.

3 Sometimes it is convenient to use a *weakening rule* in resolution. This allows the proof to add variables in any clause. In particular if you know that $(x_1 \vee x_2)$ is true the you conclude that $(x_1 \vee x_2 \vee x_i)$ is true for any i (or, of course $(x_1 \vee x_2 \vee \bar{x}_i)$, or adding two or more literals). Another useful concept is that of a *restriction* which fixes the value of any variable to a constant.

Suppose F is a non-satisfiable CNF-formula and you have a resolution proof φ of this fact. The restriction, F' of setting x_1 to true of the formula F is obtained by removing any clause that contains x_1 and removing $\bar{x}_1$ from any clause where it appears. Clauses not containing x_1 or its negation are just copied from F to F'.

3a (2p) Prove that F' is not satisfiable.

3b (10p) Prove that there is resolution proof of F' that uses no more steps than the proof φ of F. Here it might be convenient to use the weakening rule.

3c (7p) Prove that you can always get rid of any application of the weakening rule without increasing the number of steps in the proof.