SIMPLE FEM-DERIVATION

We want to derivae the FEM-formulation for the wave equation in system form (starting indexing from 0 to correspond with DOLFIN):

$$\dot{u}_0 - u_1 = 0 \tag{1}$$
$$\dot{u}_1 - u_0'' = 0 \tag{2}$$

To make the derivation easier, we first apply time-stepping (Trapezoid):

$$u_0^{n+1} - u_0^n - \frac{1}{2}ku_1^{n+1} - \frac{1}{2}ku_1^n = 0 \tag{3}$$
$$u_1^{n+1} - u_1^n - \frac{1}{2}ku_0^{n+1\prime\prime} - \frac{1}{2}ku_0^{n\prime\prime} = 0 \tag{4}$$

We define u as a vector $u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ and define the equation/residual $R(u)$ to be solved by the FEM:

$$R_0(u) = u_0^{n+1} - u_0^n - \frac{1}{2}ku_1^{n+1} - \frac{1}{2}ku_1^n = 0 \tag{5}$$
$$R_1(u) = u_1^{n+1} - u_1^n - \frac{1}{2}ku_0^{n+1\prime\prime} - \frac{1}{2}ku_0^{n\prime\prime} = 0 \tag{6}$$

with $R(u) = \begin{pmatrix} R_0(u) \\ R_1(u) \end{pmatrix}$.

We define a piecewise linear basis function ϕ_0 and ϕ_1 for each equation and also put them in a vector $\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$.

We then recall the finite element method as requiring that:

$\int_a^b R(u) \cdot \phi dx = 0$

for the basis functions ϕ.

We can now just expand our definitions to see the details of the FEM formulation:

$$\int_a^b R(u) \cdot \phi dx = 0 \Rightarrow \tag{7}$$

$$\int_a^b R_0(u)\phi_0 + R_1(u)\phi_1 dx = 0 \Rightarrow \tag{8}$$

$$\int_a^b (u_0^{n+1} - u_0^n - \frac{1}{2}ku_1^{n+1} - \frac{1}{2}ku_1^n)\phi_0 + (u_1^{n+1} - u_1^n - \frac{1}{2}ku_0^{n+1\prime\prime} - \frac{1}{2}ku_0^{n\prime\prime})\phi_1 dx = 0 \tag{9}$$

What is left to do is apply integration by parts on the terms with second derivatives, and apply the boundary conditions by using the κ formulation. We also move all the terms with a $n+1$ superscript (the unknowns) to the left hand side and denote it `a`, and move the rest of the terms to the right hand side and denote it `L`.

You should now be able to see a one-to-one correspondence with your derived formulation and the example formulation in `wave1D.py`.

A similar derivation with small modifications apply to `conv1D.py`.