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Tentamen i Kursen 2D1225
Numerisk Behandling av Differentialekvationer I
Saturday 2006-12-16 kl 8–13

Result will be ready before January 10th 2007. Next exam Jan 16th 8-13.

No means of help allowed

P0. State the number of bonus credit (max 3 cr) you have achieved from the labs.

(3) 1. How does the stability properties of an ODE-system $\dot{u} = Au$ depend on the eigenvalues of the $n \times n$-matrix A? Investigate if the following two matrices give stable solutions:

$$A = \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 9 \\ 0 & 0 \end{pmatrix}$$

(4) 2. Derive a second order difference approximation to $y'(a)$ using the values $y(a + h/2)$, $y(a + h)$ and $y(a + 2h)$. Verify the order of your approximation.

(3) 3. The stability region of a method solving an IVP is based on the test equation $\dot{y} = qy$, where q is a complex number. It is known that for Euler's explicit method the stability region is $|1 + hq| \leq 1 \quad (1)$. - Sketch the stability region for explicit Euler.
Another method for IVPs based on the formulation $\dot{y} = f(x, y)$ is $y_{k+1} = y_k + hk_2$, where $k_1 = f(x_k, y_k)$, $k_2 = f(x_k + h/2, y_k + hk_1/2)$ Derive the inequality of type (1) defining the stability region for this method.

(2) 5. What is meant by 'fill-in' when solving a large sparse linear system of equations $Ax = b$ with a method based on Gaussian elimination?

(2) 6. When using the FEM in 2D with linear triangle elements the FEM-solution $\tilde{u}(x, y)$ is obtained. The values of $\tilde{u}$ at the three corner points of a triangle is shown below. What is the FEM-solution at the origin, i.e. what is $\tilde{u}(0, 0)$?

(4) 7. The solution of the hyperbolic PDE $u_t + cu_x = 0$, $c > 0$, the advection equation, is constant along certain curves in the (x, t)-plane. What are these curves called? Give the analytic expression of these curves for the advection equation. How are these curves changed if c is modified to be t-dependent: $c(t) = c_0 + c_1 t$, $c_0 > 0, c_1 > 0, t > 0$?

(2) 8. What is meant by periodic boundary conditions?

(4) 9. The following PDE models the cooling of a hot sphere made of metal

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial T}{\partial r}), \quad 0 \leq r \leq R, \quad t \geq 0 \tag{1}$$

The initial condition states that the sphere has the same temperature T_{init} at time $t = 0$ i.e.

$$T(r, 0) = T_{init}$$

The boundary conditions of the problem are

$$\frac{\partial T}{\partial r}(0, t) = 0 \quad k\frac{\partial T}{\partial r}(R, t) = -\beta(T(R, t) - T_{out})$$

The units of the variables and parameters of this problem are:

T, T_{out}, T_{init} $[K]$
r, R $[m]$
t $[s]$
κ $[m^2/s]$, thermal diffusivity
k $[J/(m\,K)]$, conductivity
β $[J/(m^2\ K)]$, heat transfer coefficient

By a proper scaling of the variables the number of parameters can be reduced to only one parameter a:

$$\frac{\partial u}{\partial \tau} = \frac{1}{x^2}\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x}), \quad u(x, 0) = 1, \quad \frac{\partial u}{\partial x}(0, \tau) = 0, \frac{\partial u}{\partial x}(1, \tau) = au(1, \tau) \qquad (2)$$

Find this scaling and show that it gives as result the dimensionless PDE problem given above. What is the algebraic relation between a and the original parameters? Show also that a is dimensionless.

Hint: A proper scaling for T is $T = T_{out} + (T_{init} - T_{out})u$

10. The scaled version of the problem given above, i.e. eqn (2) is to be solved numerically with the Method of Lines.

(2) a) Introduce a discretization of the x-axis. Put suitable indices to the grid-points and give the relation between the stepsize h and the number N of grid-points.

(2) b) With the grid given in a) use the Method of Lines to discretize the x-variable in the PDE to a system of ODE's. Be careful and state for which gridpoints an ODE can be formulated, in particular consider the discretized version at $x = 0$.

(2) c) Discretize the boundary conditions of the problem.

(4) d) Set up the resulting ODE-system and the initial values.

(1) e) What is the analytical solution of the stationary problem?