NADA, KTH, Lennart Edsberg

Tentamen i Kursen DN2225
Numerisk Behandling av Differentialekvationer I
Wednesday 2007-12-19 kl 14–19

No means of help allowed. To pass 10 credits of max 27 is needed, provided enough credits have been achieved at the labs (at least 14) and the project (at least 1).

(2) 1. What is meant by an $n \times n$ matrix A being diagonalizable? Investigate if the following two matrices are diagonalizable:

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$$

2. Euler's explicit method, stability and accuracy

(1) a) What is the stability region for the explicit Euler method?

(2) b) Assume we want to use explicit Euler to solve the differential equation

$$y'' + 2y' + 5y = 0, \; y(0) = 1, y'(0) = 0$$

Compute the largest stepsize h_{max} that can be used if we want a stable numerical solution.

(1) c) If we want an accurate solution, the stepsize must be smaller than h_{max}. Assume we want to calculate $y(1)$ with explicit Euler using constant stepsize h. With $h = 0.1$ the error is estimated to be 10^{-2}. What constant stepsize is needed to achieve the error 10^{-3}?

(3) 3. The one dimensional version of Navier-Stokes equation is

$$\rho(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 v}{\partial x^2} + \rho g$$

Introduce dimensionless variables: $u = v/V, z = x/L, q = (p - p_0)/(\rho V^2), \tau = tV/L$, where the parameters are: V is a characteristic velocity, L is a characteristic length, p_0 is a reference pressure, ρ is the density, μ the viscosity coefficient and g the earth gravity. All parameters are constant.
Express the original differential equation in the dimensionless variables and present the differential equation using the dimensionless parameters Reynold's number $Re = LV\rho/\mu$ and Froude's number $Fr = V^2/gL$.

(3) 4. Derive a difference approximation formula for $y''(0)$ being of highest possible order using the values $y(-h)$, $y(h)$ and $y(2h)$. What is the order?

5. Given the parabolic PDE problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = x(1-x), \quad u(0,t) = 0, u(1,t) = 1$$

and the ansatz function

$$\tilde{u}(x,t) = \sum_{j=1}^{N-1} u_j(t)\varphi_j(x) + \varphi_N(x)$$

where the interval $[0,1]$ is discretized: $x_j = jh, j = 0,1,2,\ldots,N-1,N, \quad Nh = 1$. The basis functions $\varphi_j(x)$ are the piecewise linear functions known as the "roof" functions.

(1) a) Present a graph showing what a roof function is. - Verify that $\tilde{u}(x,t)$ satisfies the boundary conditions of the PDE problem.

(4) b) When Galerkin's method is applied to solve the problem an ODE-system is obtained:

$$M\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{b}, \quad \mathbf{u}(0) = \mathbf{u}_0$$

What are the dimensions of M,A and $\mathbf{b}$? Give the elements of M, A and $\mathbf{b}$ in terms of the basis functions, i.e. the integrals must not be computed. What structure will the matrices M and A have?

(2) 6. What is meant by Cholesky factorization of an SPD (symmetric positive definite) matrix A? What is meant by "fill-in" when factorizing a sparse SPD matrix with this method?

7. Consider the following coupled system of one partial and one ordinary differential equation:

$$\frac{\partial^2 c_1}{\partial r^2} + \frac{1}{r}\frac{\partial c_1}{\partial r} - \frac{\partial c_1}{\partial z} = 0, \qquad R_2 < r < R_3, \quad z > 0 \tag{1}$$

$$\frac{d^2 c_2}{dr^2} + \frac{1}{r}\frac{dc_2}{dr} + c_2 = 0, \qquad 0 < R_1 < r < R_2 \tag{2}$$

with boundary conditions:

$$\frac{\partial c_1}{\partial r}(R_3, z) = 0, \frac{\partial c_1}{\partial r}(R_2, z) = \frac{dc_2}{dr}(R_2, z), c_1(R_2, z) = c_2(R_2, z), \frac{dc_2}{dr}(R_1, z) = 0$$

and the initial condition: $c_1(r, 0) = a$

Observe that c_2 is independent of z in the ODE but will be z-dependent through the coupling with c_1 through the boundary conditions!

a) (1) Of which type is the partial differential equation? Elliptic, parabolic or hyperbolic?

b) (5) Discretize the r-axis with stepsize h, so that $Mh = R_2 - R_1$ and $Nh = R_3 - R_2$.

Use the method of lines to discretize the partial differential equation (1). Use central difference approximations for the derivatives with respect to r. Set up the ODE-system

and describe the matrix A_1 and the vectors $\mathbf{b}_1, \mathbf{c}_{10}$ in

$$\frac{d\mathbf{c}_1}{dz} = A_1\mathbf{c}_1 + \mathbf{b}_1, \quad \mathbf{c}_1(0) = \mathbf{c}_{10}$$

The vector $\mathbf{b}_1$ will contain components of the c_2-variable (coming from the boundary conditions)

c) (3) Discretize the ordinary differential equation (2). Use central difference approximations for the derivatives. Set up the system of algebraic equations $A_2\mathbf{c}_2 = \mathbf{b}_2$, where $\mathbf{b}_2$ contains a few components of the c_1-variable (coming from the boundary conditions).

d) (2) Suggest a numerical method (not just a name, give the formula also) for solving this kind of system being on the form (differential-algebraic)

$$\frac{d\mathbf{c}_1}{dt} = \mathbf{f}_1(\mathbf{c}_1, \mathbf{c}_2), \mathbf{c}_1(0) = \mathbf{c}_{10}$$

$$0 = \mathbf{f}_2(\mathbf{c}_1, \mathbf{c}_2)$$

If you have not solved b) or c) you can still suggest an answer to this part, remembering that the ODE-system is stiff.