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Tentamen i Kursen DN2225
Numerisk Behandling av Differentialekvationer I
Thursday 2008-12-18 kl 8–13

No means of help allowed. To pass 13 credits of max 27 is needed, provided enough credits have been achieved at the labs (at least 18) and the project (at least 1).

1. Assume that the $n \times n$ matrix A is diagonalizable with eigenvalues in the diagonal matrix D and the corresponding eigenvectors as columns in the matris S.

(1) a) What is the definition of the exponential matrix e^{At}?

(2) b) Calculate e^{At} when

$$A = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix}$$

(2) c) In the definition of e^{At}, nothing is said about the lengths of the eigenvectors of A. Verify that e^{At} does not depend on the lengths of the eigenvectors.

2. Given a differential equation and a corresponding approximating difference equation

$$\frac{d^2u}{dx^2} + \lambda \frac{du}{dx} = 0, \qquad \frac{\nabla^2 u_n}{h^2} + \lambda \frac{\nabla u_n}{h} = 0$$

where $\nabla u_n = u_n - u_{n-1}$ and $\nabla^2 u_n = \nabla(\nabla u_n)$. The parameter λ is a complex number and the stepsize $h > 0$.

(1) a) For which values of λ is the differential equation stable?

(2) b) What is the characteristic equation of the differential equation and what is the characteristic equation of the difference equation given above?

(2) c) For which values of $h\lambda$ is the difference equation stable?

(3) 3. Derive an unsymmetric difference approximation formula $y'''(0) = ay(-h) + by(0) + cy(h) + dy(2h) + O(h^p)$ being of highest possible order p. It is enough to present the system of equations defining a, b, c and d. For full credit also compute p.

(2) 4. What is meant by an upwind scheme for solving the advection equation? ($a > 0$)

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Which are the characteristics when $a = 2$?

5. Given the parabolic problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad u(0,t) = u(1,t) = 0, \quad 0 \le x \le 1, t > 0$$

Discretization of the problem will lead to an initial value problem for a linear system of ODE's.

a) **(2)** Present the Galerkin method with an ansatz $\tilde{u}(x,t) = \sum_{k=1}^{n} a_k(t)\varphi_k(x)$ and describe the matrices and vectors in the ODE-system for the coefficients $a_k(t)$.

b) **(2)** Present the Method of Lines (MoL), describe the matrices and vectors in the ODE-system and show the recursion formula when the trapezoidal method is used to solve the ODE-system.

6. The following PDE models the temperature $T(z,r)$ of a liquid being transported in a cylindrical tube with radius R and length L. The liquid is heated by the wall and the heat energy is transported by convection in the z-direction and diffusion in the r-direction.

$$D_r\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) - v\frac{\partial T}{\partial z} = 0, \quad 0 \le r \le R,\ 0 \le z \le L \tag{1a}$$

The boundary conditions (BC) are:

$$T(R,z) = T_0, \quad \frac{\partial T}{\partial r}(0,z) = 0,\ 0 \le z \le L, \qquad T(r,0) = 0,\ 0 \le r \le R \tag{1b}$$

In the equations above, v, D_r and T_0 are positive constants.

(1) a) What kind of PDE is this, parabolic, elliptic or hyperbolic?

(4) b) Introduce a suitable notation and a suitable grid with constant stepsize in the r-direction to formulate the Method of Lines (MoL) for the problem. Be careful to mention which difference approximations have been used and their order of accuracy. When some of the BCs have been discretized and inserted into the ODE-system, it will be of the form

$$\frac{d\mathbf{T}}{dz} = A\mathbf{T} + \mathbf{b} \tag{2}$$

What is A and $\mathbf{b}$ for this problem? Finally what is the initial vector of the ODE-system (2)?

(3) c) Assume that the problem is changed so that heat is transported with diffusion also in the z-direction, i.e. the PDE is changed to

$$D_r\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + D_z\frac{\partial^2 T}{\partial z^2} - v\frac{\partial T}{\partial z} = 0 \tag{3}$$

where D_z is a positive constant. What type of PDE is (3), parabolic, elliptic or hyperbolic? Suggest a modification of the BCs given in (1b) suitable for this problem (hint: one BC is missing). If the PDE-problem (3) is treated numerically, what kind of numerical method should be used after discretization? (You don't have to present the discretization but just give a brief description of how the problem (3) turns into a numerical problem and what method should be used to solve the problem efficiently).